

Inverse scattering using a single incident wave

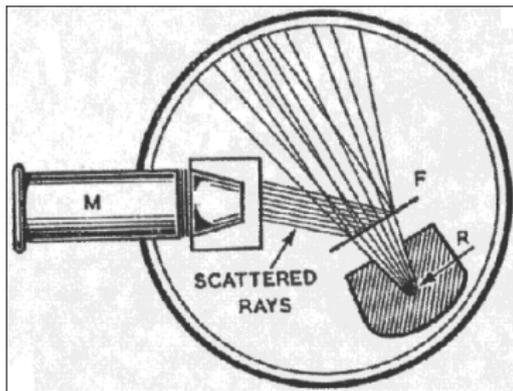
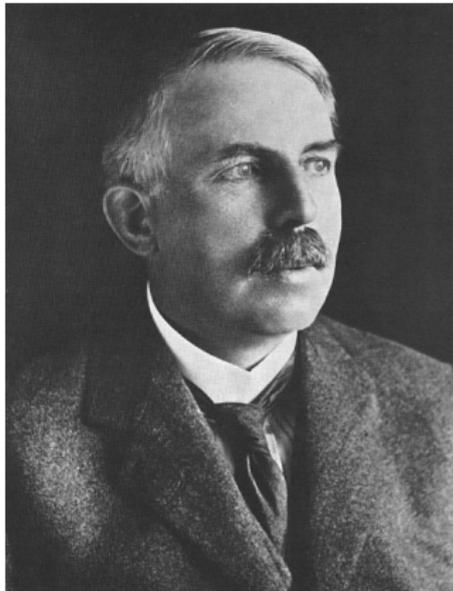
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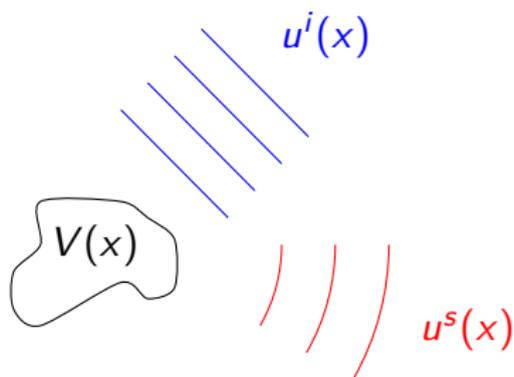
2nd East Asia Section of IPIA
Young Scholars Symposium
National Center for Theoretical Sciences,
National Taiwan University

November 5 – 6, 2016

Scattering theory



Scattering theory



The total wave u satisfies

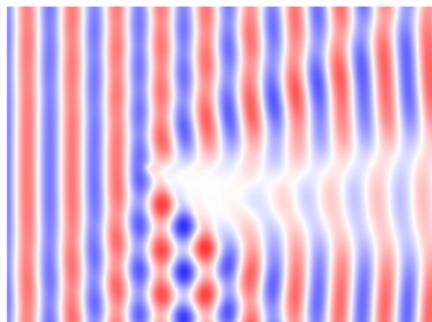
$$(\Delta + k^2(1 + V))u = 0,$$

V models a **perturbation** of the background,

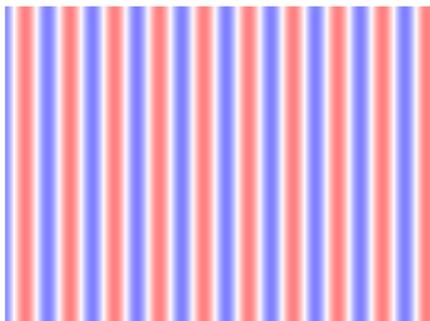
$$u = u^i(x) + u^s(x)$$

\uparrow incident wave \leftarrow scattered wave

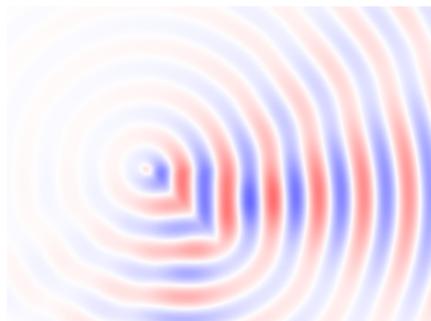
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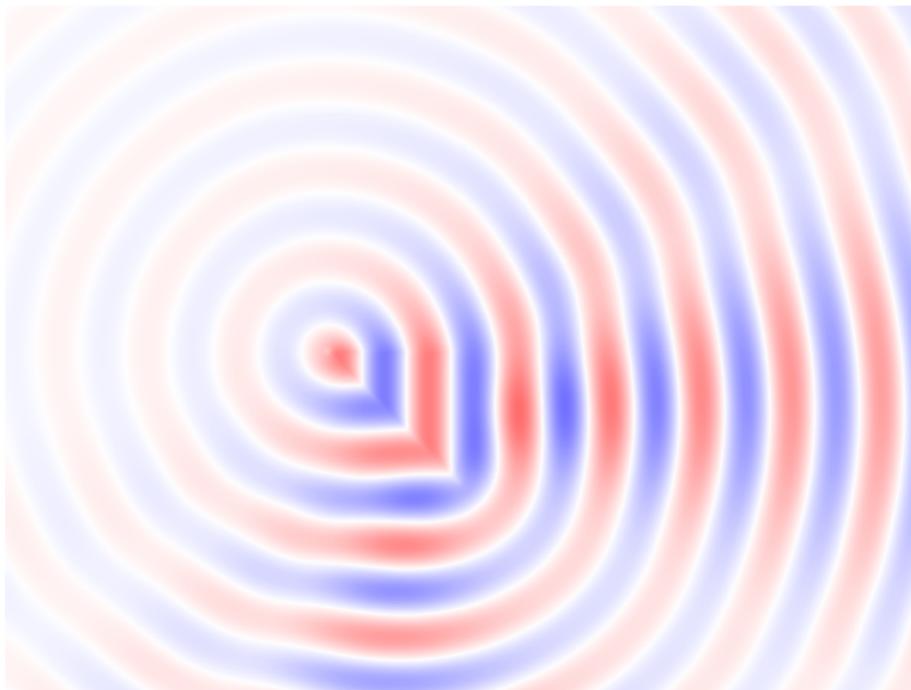
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Mathematical scattering theory: measurements



Measurement: A_{u^i} is the **far-field pattern** of the scattered wave

$$u(x) = u^i(x) + \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i} \left(\frac{x}{|x|} \right) + \mathcal{O} \left(\frac{1}{|x|^{n/2}} \right)$$

Inverse problem

Given the map $u^i \mapsto A_{u^i}$, recover V or its support Ω .

Early methods (< 85')

- ▶ optimization and minimization methods

Inverse problem

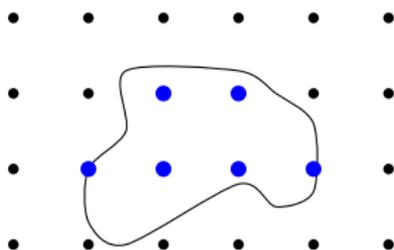
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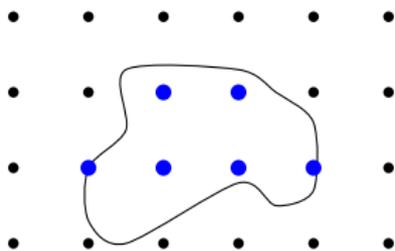
- ▶ optimization and minimization methods

Sampling methods

- ▶ gives condition on measurements for $x \in \text{supp } V$
- ▶ compared to before: *fast! works reliably!*

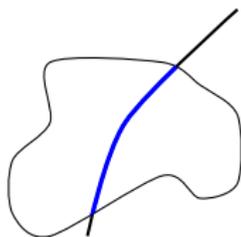


Sampling methods



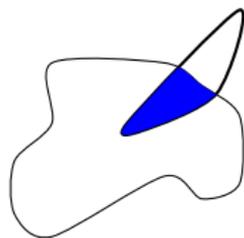
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Sampling methods



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Sampling methods



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- ▶ 98 Ikehata: probing method (curve)
- ▶ ... Luke, Potthast, Sylvester, Kusiak: range test, no response test (sets)

Factorization method

Sampling methods gave only¹ *sufficient* conditions for $x \in \text{supp } V$.

¹except Ikehata's probing method

Factorization method

Kirsch 90's, Grinberg 00's: factorization method. Gives *necessary and sufficient* conditions.

Factorization method

Idea:

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta), \quad g \in L^2(\mathbb{S}^{n-1})$$

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_g \left(\frac{x}{|x|} \right) + \mathcal{O} \left(\frac{1}{|x|^{n/2}} \right)$$

the far-field operator

$$F : L^2(\mathbb{S}^{n-1}) \rightarrow L^2(\mathbb{S}^{n-1}), \quad Fg = A_g$$

is factored

$$F = G T G^*$$

G compact, T isomorphism. The range of G can be characterized and gives $\text{supp } V$.

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$$\begin{aligned}(\Delta + k^2)v &= 0, & \Omega \\(\Delta + k^2(1 + V))u &= 0, & \Omega \\u - v &\in H_0^2(\Omega)\end{aligned}$$

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k^2 is an interior transmission eigenvalue (ITE)

Some ITE history

- ▶ 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- ▶ 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE

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- ▶ 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- ▶ 08' **Päivärinta–Sylvester**: existence for general scatterers
- ▶ 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- ▶ 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- ▶ 10'+: EXPLOSION OF INTEREST

Interior transmission eigenvalues VS sampling methods

Recall: $\ker F \neq \{0\} \implies k^2$ ITE

Sampling method users avoid ITE's

Are they too careful?

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 $\implies \ker F \neq \{0\}$
- ▶ Regge, Newton, Sabatier, Grinevich, Manakov, Novikov
50's – 90's: radial potentials transparent at a fixed k^2 i.e.
 $\implies \ker F = L^2(\mathbb{S}^{n-1})$

Corner scattering

- ▶ B.–Päivärinta–Sylvester 14: $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ always scatters, *despite having interior transmission eigenvalues*

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$$k^2 \text{ ITE and } \ker F = \{0\}$$

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if $(\Delta + k^2(1 + V))u_0 = 0$ near $\text{supp } V$.

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In simple case

$$u^i(x) = 1 + u_r^i(x)$$

$$u_0(x) = e^{\rho \cdot x}(1 + \psi(x))$$

$$V(x) = \chi_{[0, \infty[^n}(x)(\varphi(0) + \varphi_r(x))$$

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Hölder estimates give

$$C |\varphi(0)| |\rho|^{-n} \leq \left| \varphi(0) \int_{[0, \infty[^n} e^{\rho \cdot x} dx \right| \leq C |\rho|^{-n-\delta}$$

if $\|\psi\|_p \leq C |\rho|^{-n/p-\varepsilon}$.

Newer corner scattering results

- ▶ Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- ▶ Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
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Injectivity of support probing:

Theorem

Let P, P' be convex polygons and $V = \chi_P \varphi$, $V = \chi_{P'} \varphi'$ for admissible functions φ, φ' . Then

$$P \neq P' \implies \ker F_V \cap \ker F_{V'} = \{0\}.$$

Any *single* incident wave determines P in the class of polygonal penetrable scatterers.

Stability of polygonal scatterer probing

Theorem (B., Liu, preprint)

Let u^i be an admissible incident wave and let $V = \chi_P \varphi$,
 $V' = \chi_{P'} \varphi'$ be admissible. If

$$\|A_{u^i} - A'_{u^i}\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C(\ln \ln \|A_{u^i} - A'_{u^i}\|_2^{-1})^{-\eta}$$

for some $\eta > 0$.

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Probing impenetrable scatterers with few waves: J. Li, H. Liu, M. Petrini, L. Rondi, J. Xiao, Y. Wang ...

Proof structure

- ▶ Quantify everything in corner scattering proofs
- ▶ Assume total wave does not vanish in domain of interest
- ▶ Propagate smallness from ∞ to $P \cup P'$

Far-field to near-field to boundary

- ▶ A quantitative version of Rellich's theorem + unique continuation

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Proposition

Let u^s and u'^s be the scattered waves caused by u^i . If $Q = \text{ch}(P \cup P')$ and $\|u_\infty^s - u_\infty'^s\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$ then

$$\sup_{\partial Q} |u - u'| + |\nabla(u - u')| \leq C \left(\ln \ln \|u_\infty^s - u_\infty'^s\|_2^{-1} \right)^{-1/2}.$$

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- ▶ $(\Delta + k^2)(u^s - u'^s) = f$ with f supported on $P \cup P'$.
- ▶ second logarithm arises from continuing $u^s - u'^s$ from the set where $f \equiv 0$ to its boundary by smoothness.

Boundary to neighbourhood of corner

Let $Q = \text{ch}(P \cap P')$, $Q_h = Q \cap B(x_c, h)$. If u_0 a CGO solution for V then

$$k^2 \int_{Q_h} V u' u_0 dx = \int_{\partial Q_h} (u_0 \partial_\nu (u' - u) - (u' - u) \partial_\nu u_0) d\sigma.$$

Estimates

$$k^2 \int_{Q_h} \nabla u' u_0 dx = \int_{\partial Q_h} (u_0 \partial_\nu (u' - u) - (u' - u) \partial_\nu u_0) d\sigma.$$

Split LHS as before and use $u' \neq 0$ everywhere in Q_h . CGO and Hölder estimates give

$$C \leq |\rho|^n \left| \int_{\mathfrak{A}} e^{\rho \cdot x} dx \right| \leq h^{-1} |\rho|^{-\delta} + |\rho|^3 (\ln \ln \|u_\infty^s - u_\infty'^s\|_2^{-1})^{-1/2}.$$

The claim follows.

Future work

- ▶ quantify corner scattering results — should work for arbitrary incident waves
- ▶ probe value of potential at corner
- ▶ probe piecewise constant potentials

Thank you for your attention!