

# Inverse spectral problem on discrete graphs

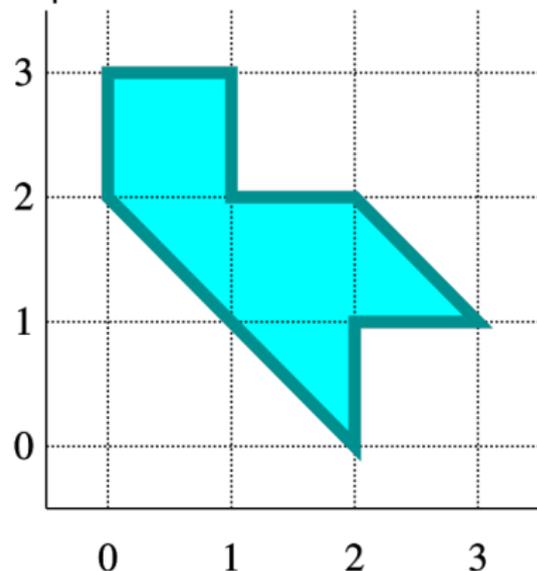
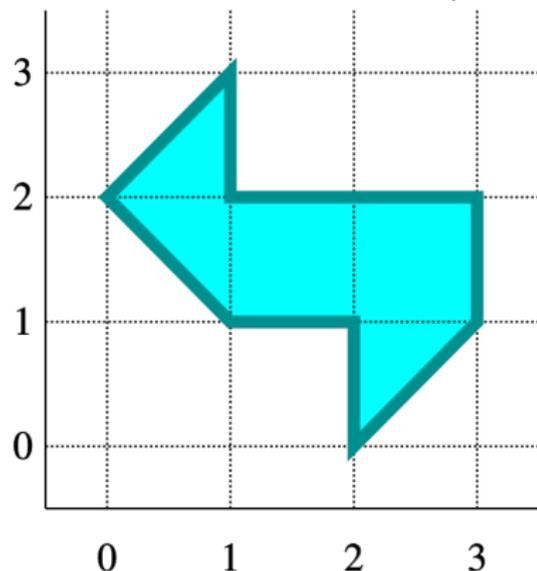
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## Can one hear the shape of a drum?

Mark Kac 1966 & Gordon, Webb, Wolpert 1992



Drumhead shapes whose vibration frequencies are the same. Same Dirichlet Laplacian eigenvalues, different domains.

## An inverse spectral problem for a manifold

Let  $M$  be a Riemannian manifold with boundary  $\partial M$  and metric  $g$ ,

$$\Delta_g u = \sum_{j,k=1}^n |g(x)|^{-1/2} \frac{\partial}{\partial x^j} (|g(x)|^{1/2} g^{jk}(x) \frac{\partial}{\partial x^k} u(x)). \quad (1)$$

The eigenvalues  $\lambda_j$  and orthonormal eigenfunctions  $\varphi_j(x)$  satisfy

$$H\varphi_j(x) = \lambda_j \varphi_j(x), \quad \text{for } x \in M, \quad H = -\Delta_g + q(x), \quad (2)$$

$$\partial_\nu \varphi_j(x) = 0 \quad \text{for } x \in \partial M. \quad (3)$$

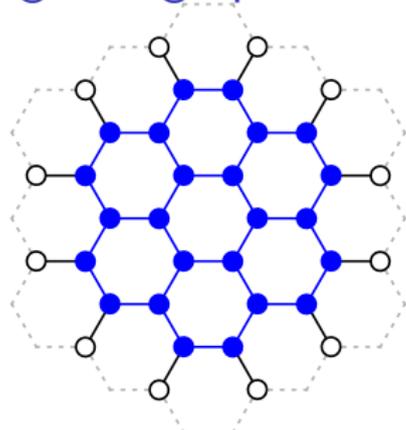
**Inverse problem:** Suppose we are given the boundary spectral data

$$(\partial M, (\lambda_j, \varphi_j|_{\partial M})_{j=1,2,\dots}) \quad (4)$$

Can we determine  $(M, g)$  and  $q$ ?

This inverse problem is equivalent to inverse problems for the wave and Schrödinger equations

## Weighted graph with boundary



We consider a finite graph with vertices  $X = G \cup B$  and edges  $E$ . We call  $G$  the interior nodes and  $B = \partial G$  the boundary nodes.

Weights:  $g_{xy}$  for  $(x, y) \in E$  and  $\mu_x$  for  $x \in X$ .

For a function  $u : G \cup \partial G \rightarrow \mathbb{R}$ , the graph Laplacian  $\Delta_G$  on  $G$  is

$$(\Delta_G u)(x) = \frac{1}{\mu_x} \sum_{y \sim x} g_{xy} (u(y) - u(x)), \quad x \in G, \quad (5)$$

and the Neumann boundary value  $\partial_\nu u$  of  $u$  is

$$(\partial_\nu u)(z) = \frac{1}{\mu_z} \sum_{x \sim z, x \in G} g_{xz} (u(x) - u(z)), \quad z \in \partial G. \quad (6)$$

For the combinatorial Laplacian,  $g_{xy} = 1$  and  $\mu_x = 1$ .

## An inverse spectral problem for a graph

Let  $(G \cup \partial G, E)$  be a graph with weights  $g$  and  $\mu$ . Let  $q : G \rightarrow \mathbb{R}$  be a potential function.

The eigenvalues  $\lambda_j$  and orthonormal eigenfunctions  $\varphi_j(x)$  satisfy

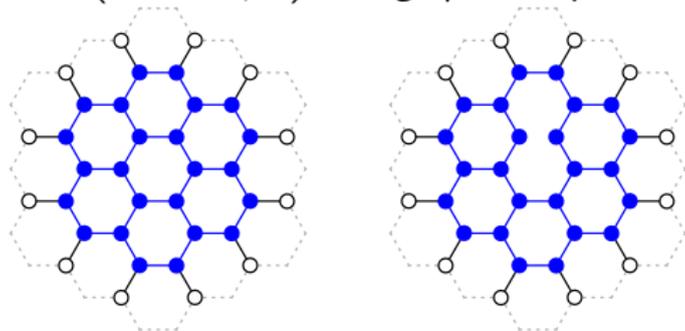
$$H\varphi_j(x) = \lambda_j\varphi_j(x), \quad \text{for } x \in G, \quad H = (-\Delta_G + q(x)), \quad (7)$$

$$\partial_\nu\varphi_j(x) = 0 \quad \text{for } x \in \partial G. \quad (8)$$

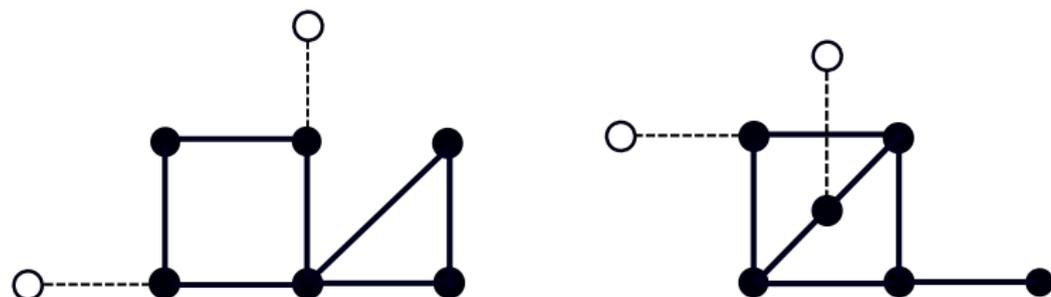
**Inverse problem:** Suppose we are given the boundary spectral data

$$(\partial G, (\lambda_j, \varphi_j|_{\partial G})_{j=1,2,\dots,|G|}) \quad (9)$$

Can we determine  $(G \cup \partial G, E)$  and  $g$ ,  $\mu$  and  $q$  on  $G$ ?



## Counterexample for the unique solvability of IP

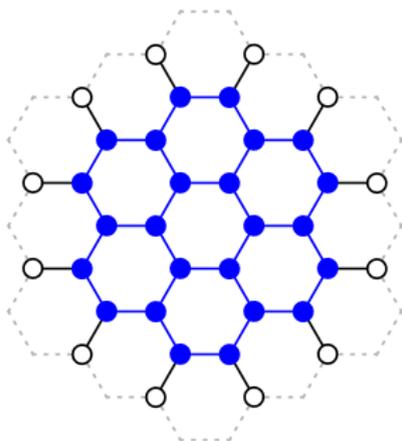


Two graphs on which the combinatorial Laplacian has the same eigenvalues and the boundary values of the eigenfunctions. The white vertices are boundary nodes and the black vertices are interior nodes (B.-Isozaki-Lassas-Lu 2021).

## Definition (Paths and metric)

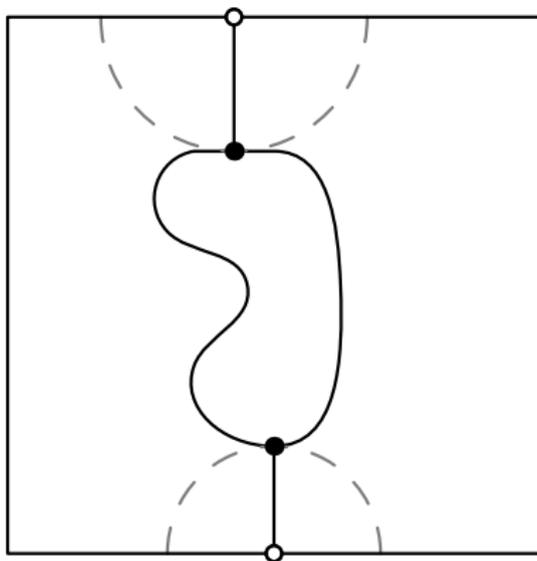
Let  $x, y \in G \cup \partial G$ . A *path* from  $x$  to  $y$  is a sequence of vertices  $v_0, v_1, \dots, v_J$  such that  $v_0 = x$ ,  $v_J = y$  and  $v_j \sim v_{j+1}$ . The length of the path is  $J$ .

The *distance*  $d(x, y)$  is the minimal length of a path connecting  $x$  and  $y$



## Definition

Let  $S \subset G$ . We say a point  $x_0 \in S$  is an *extreme point* of  $S$ , if there exists  $z \in \partial G$  such that  $x_0$  is the unique nearest point in  $S$  from  $z$ .



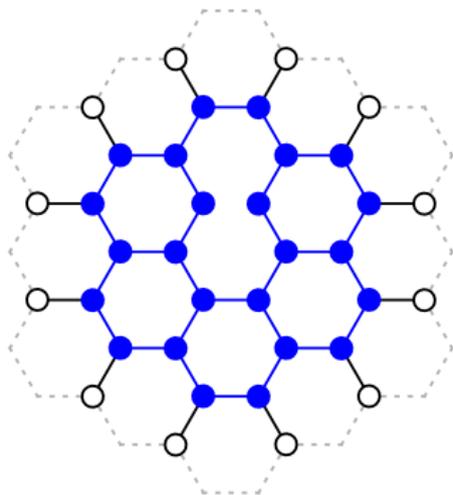
**Figure:** Any compact subset of the unit square which has at least two points has at least two extreme points.

We impose the following assumptions on the graph  $(G, \partial G, E)$ .

Two-Points Condition:

- (1) For any subset  $S \subset G$  with at least 2 points, there exist at least two extreme points.
- (2) Each boundary vertex is connected to only one interior vertex.

This condition is valid for trees, when all vertices of degree one are boundary nodes, and for perturbations of the standard lattices.



## Theorem (B.-Isozaki-Lassas-Lu 2021)

Let  $(G, \partial G, E, \mu, g)$  and  $(G', \partial G', E', \mu', g')$  satisfy the Two Points Condition. Let  $q, q'$  be potential functions on  $G, G'$ .

Assume that  $\partial G$  and  $\partial G'$  can be identified using a bijective map and the boundary spectral data of these graphs are the same.

Then there is a bijection  $\Phi : G \cup \partial G \rightarrow G' \cup \partial G'$  such that

$$x_1 \sim x_2 \text{ if and only if } \Phi(x_1) \sim' \Phi(x_2).$$

Moreover, if we use  $\Phi$  to identify the graphs  $G$  and  $G'$ , then

- (1) If  $\mu = \mu'$ , then  $g = g'$  and  $q = q'$ .
- (2) If  $q = q' = 0$ , then  $\mu = \mu'$  and  $g = g'$ .

In other words: The observations at the boundary nodes are enough to determine the structure in the interior of the graph, even when all of the interior nodes  $G$  and the edges  $E$  are unknown.

## Theorem (B.-Isozaki-Lassas-Lu 2021)

If on the graph  $(G, \partial G, E)$  there exists  $h : G \cup \partial G \rightarrow \mathbb{R}$  such that

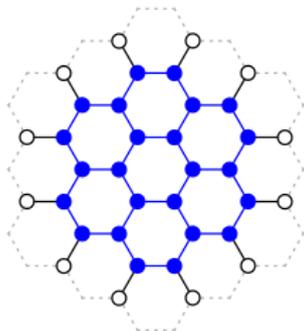
1. If  $x \sim y$  then  $|h(x) - h(y)| \leq 1$ ,
2.  $|N_{\pm}(x)| = 1$  for all  $x \in G$ , and  $|N_{\pm}(z)| \leq 1$  for all  $z \in \partial G$ ,  
where

$$N_{+}(x) = \{y \in G \cup \partial G : y \sim x, h(y) = h(x) + 1\},$$

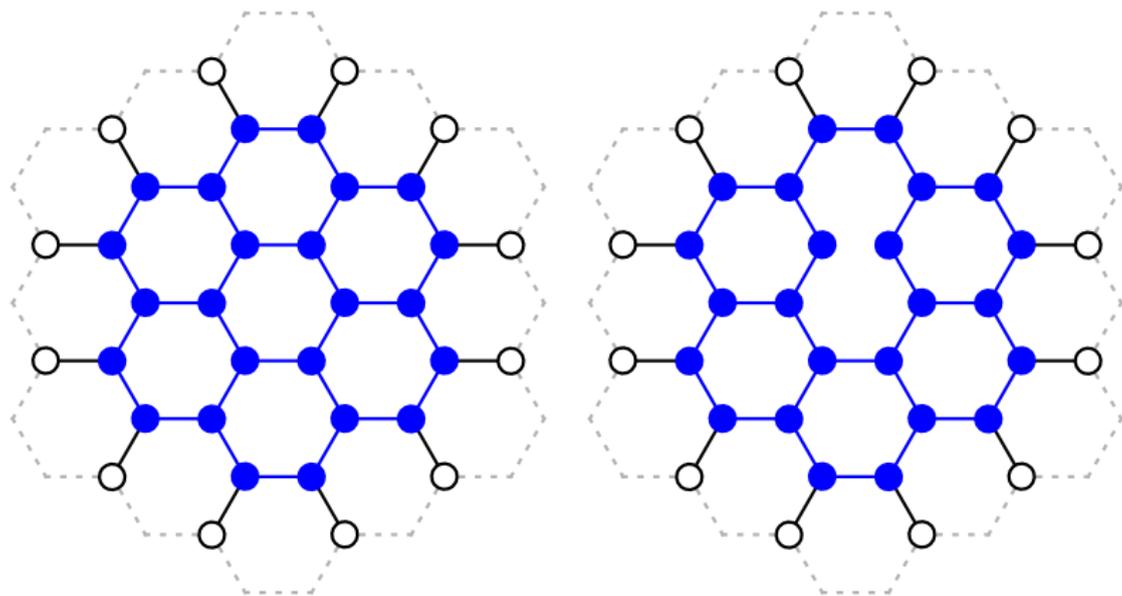
$$N_{-}(x) = \{y \in G \cup \partial G : y \sim x, h(y) = h(x) - 1\},$$

then  $(G, \partial G, E)$  satisfy the Two-Points Condition.

We call  $N_{+}(x)$  the discrete gradient of  $h$  at  $x$ , and  $N_{-}(x)$  the discrete gradient of  $-h$  at  $x$ .



## Graphs satisfying the Two Points Condition

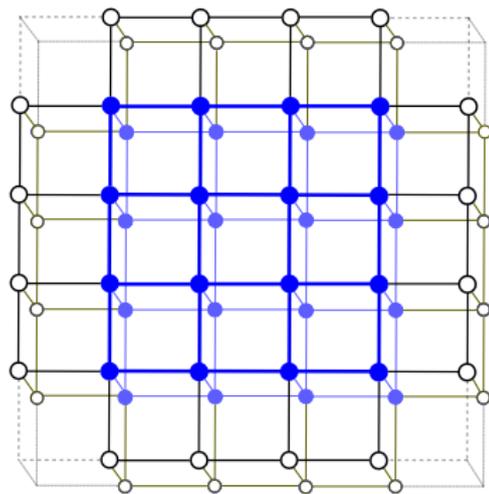


**Figure:** Finite hexagonal lattice. The white vertices are considered to be the boundary vertices for the set of the blue (interior) vertices. Also, any horizontal edges can be removed.

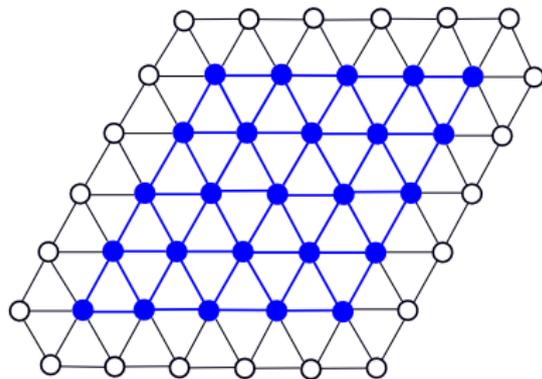
## Graphs satisfying the Two Points Condition

Following graphs satisfy the Two Points Condition when suitable connections to the boundary vertices are removed.

The white vertices are the boundary vertices; the blue vertices are the interior vertices.



**Figure:** Finite two-level square ladder, made out of two layers of square lattices.



**Figure:** Finite triangular lattice.

## Proof idea

### Lemma

Suppose  $W$  is the initial value of some wave  $u$  satisfying the wave equation

$$\begin{cases} D_{tt}u(x, t) - \Delta_G u(x, t) + q(x)u(x, t) = 0, & x \in G, t \geq 1, \\ \partial_\nu u(x, t) = 0, & x \in \partial G, t \geq 0, \\ D_t u(x, 0) = 0, & x \in G, \\ u(x, 0) = W(x), & x \in G \cup \partial G, \end{cases} \quad (10)$$

Then the spectral data and the Fourier coefficients  $\widehat{W}, \widehat{W}_0$  of the initial values  $W, W_0$  determine

- ▶ the boundary values  $u(z, t), z \in \partial G, t \geq 0,$
- ▶ inner products  $\langle u(\cdot, t), W_0 \rangle,$

Here

$$\widehat{W}(j) := \langle W, \varphi_j \rangle := \sum_{x \in G} \mu_x W(x) \varphi_j(x).$$

## Proof idea

Associate any vertex  $x_0 \in G$  with the initial value  $W_{x_0}$  for which

$$W_{x_0}(x) = \delta_{x_0}(x), \quad x \in G.$$

Let  $\mathcal{A}_0$  be the set of all initial values  $W$  which are supported at a single point in  $G$ .

Then the boundary spectral data determines the Fourier coefficients of elements of  $\mathcal{A}_0$ , i.e.

$$\widehat{\mathcal{A}}_0 = \{\widehat{W}_x : x \in G\}$$

is determined.

## Edge determination

### Lemma

Let  $\mathbb{G}$  be a finite weighted graph with boundary satisfying the two-point condition, and  $x, y \in G$ . Then  $x \sim y$  if and only if

$$\min \{t \in \mathbb{N} \mid \langle u^{W_x}(\cdot, t), W_y \rangle \neq 0\} = 2. \quad (11)$$

We can now identify points  $x, y \in G$  with Fourier coefficients  $\widehat{W}_x, \widehat{W}_y$  of single point initial values  $W_x, W_y$ . And we can use these to determine when  $\langle u^{W_x}(\cdot, t), W_y \rangle \neq 0$ .

The boundary spectral data determines if  $x \sim y$ .

## Coefficient determination

For (i), i.e.  $\mu$  is known, we determine  $g_{xy}$  by

$$\langle (-\Delta_G + q)W_x, W_y \rangle = -W_y(y) \sum_{p \sim y} g_{yp} W_x(p) \quad (12)$$

whose LHS is uniquely determined by the boundary spectral data:

$$\begin{aligned} W_x &= \sum_j \langle W_x, \varphi_j \rangle \varphi_j, & W_y &= \sum_j \langle W_y, \varphi_j \rangle \varphi_j, \\ \langle W_x, \varphi_j \rangle &= \langle W_x, \varphi_j' \rangle', & (-\Delta_G + q)\varphi_j &= \lambda_j \varphi_j. \end{aligned}$$

Similar arguments applied to  $\langle (-\Delta_G + q)W_x, W_x \rangle$  determine  $q$ .

For (ii), i.e. when  $q = 0$ , there is an eigenvalue such that  $\lambda_0 = 0$  and  $\varphi_0 = c$ . Then

$$\mu_x = \langle W_x, \varphi_0 \rangle^2 c^{-2} \quad (13)$$

is determined by the boundary spectral data.

Thank you!