Inverse problems with one measurement

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The Institute for Advanced Study,
The Hong Kong University of Science and Technology

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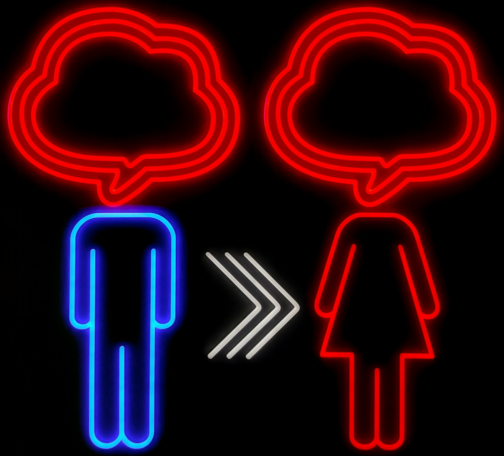
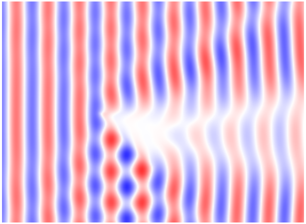


Scattering theory



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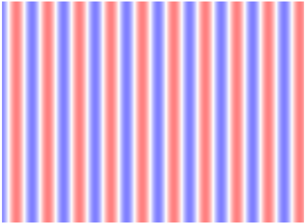

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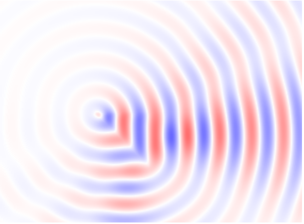
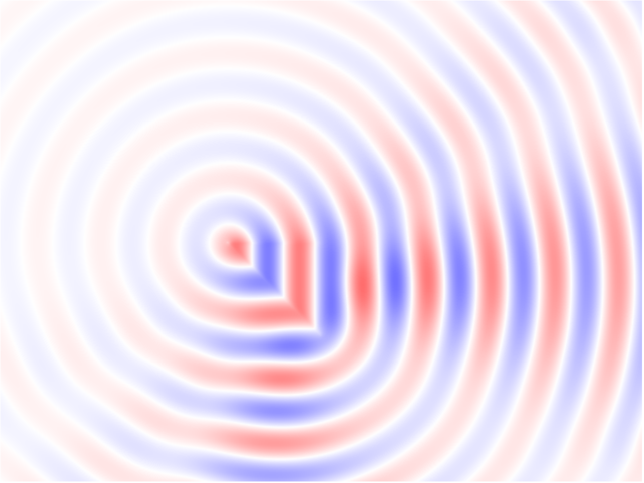


Mathematical scattering theory: measurements



Measurement: A_{u^i} is the far-field pattern of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i}\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

Inverse problems

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Solved when

- full far-field map given for all large frequencies (Saito 84),
- full far-field map given for a single frequency (Sylvester–Uhlmann 87 $n \geq 3$ + Bukhgeim 07 $n = 2$),
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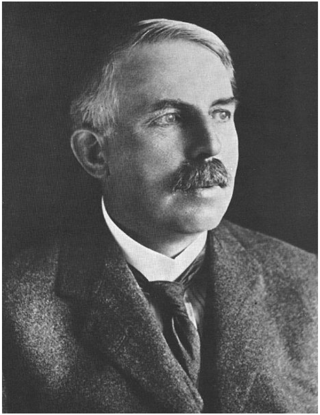
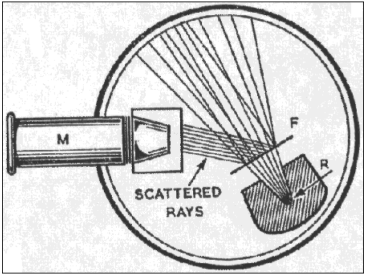
Focus on *single measurement*: A_{u^i} given only for a single u^i.

Schiffer's problem: can a single measurement determine Ω?

What about in physics?

Lord Rutherford's gold-foil experiment

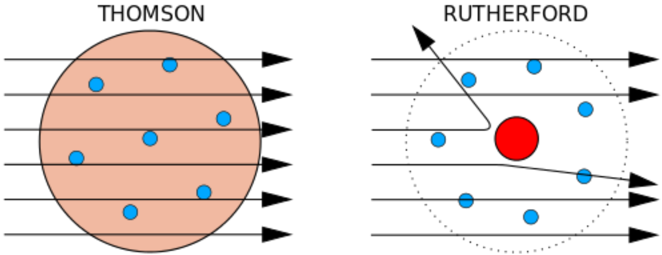




Single incident wave

Scattering theory

Rutherford experiment's conclusions



measurement + *a-priori information* = conclusion

What if the measurement gives nothing?

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However A_{u^i} can become arbitrarily small with $\|u^i\| \geq C > 0$.
(TE)

Some follow-up corner scattering results

- Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- Elschner–Hu: 3D any domain having two faces meet at an angle, and also curved edges
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Injectivity of the Schiffer's problem for polyhedra:

Theorem (HSV+EH)

Let P, P' be convex polyhedra and $V = \chi_P \varphi$, $V = \chi_{P'} \varphi'$ for admissible functions φ, φ'. Then

$$P \neq P' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i \neq 0$$

Any *single* incident wave determines P in the class of polyhedral penetrable scatterers.

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However Ikehata's enclosure method gives roughly the same!

My work while in Hong Kong

Stability of polygonal scatterer probing

Non-vanishing total wave

Theorem (B., Liu, preprint)

Let u^i be an incident wave and let $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ be admissible with $|u|, |u'| \neq 0$ in B_R. If

$$\left\| A_{u^i} - A'_{u^i} \right\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C \left(\ln \ln \left\| A_{u^i} - A'_{u^i} \right\|_2^{-1} \right)^{-\eta}$$

for some $\eta > 0$.

Lower bound for far-field pattern

Arbitrary Herglotz wave

Theorem (B., Liu, JFA 2017)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible. Then

$$\|A_{u^i}\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|, V} > 0$$

where the Taylor expansion of u^i at the corner x_c begins with P_N, and $\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)|\, d\sigma(\theta)$.

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

From apparent contradiction to inspiration

Theorem (B., Liu, + B., Li, Liu, Wang, JFA + IP 2017 + preprint)

Let the potential $V = \chi_\Omega \varphi$ be admissible. Let $v, w \neq 0$ be transmission eigenfunctions:

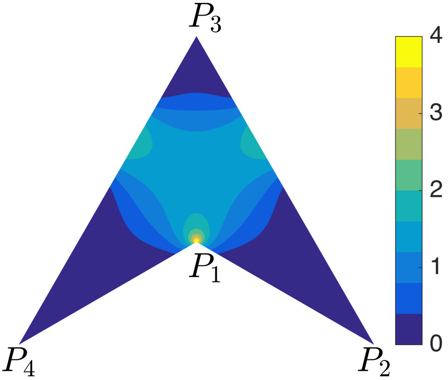
$$\begin{aligned} (\Delta + k^2)v &= 0, \qquad \Omega \\ (\Delta + k^2(1+V))w &= 0, \qquad \Omega \\ w - v &\in H_0^2(\Omega). \end{aligned}$$

Under C^α-smoothness of v near x_c, we have

$$v(x_c) = w(x_c) = 0$$

at every corner point x_c of Ω.

Transmission eigenfunction localization



Piecewise constant recovery

Injectivity of piecewise constant potential probing:

Theorem (B., Liu, preprint)

Let Σ_j, $j = 1, 2, \ldots$ be bounded convex polyhedra in an admissible geometric arrangement (think pixels/voxels) and $V = \sum_j V_j \chi_{\Sigma_j}$, $V' = \sum_j V'_j \chi_{\Sigma_j}$ for constants $V_j, V'_j \in \mathbb{C}$. Then

$$V \neq V' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i(x) = e^{ik\theta\cdot x}$$

if $k > 0$ small or $|u| + |u'| \neq 0$ at each vertex.

A *single* incident plane wave determines V in the class of discretized penetrable scatterers.

Generalizations and limitations

- unique determination of corner location *and* value

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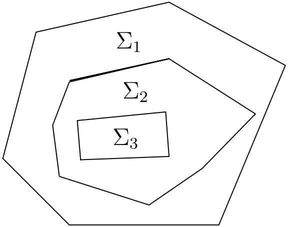
- if Σ_j not known in advance: both $(\Sigma_j)_{j=1}^{\infty}$ and $V = \sum_j V_j \chi_{\Sigma_j}$ uniquely determined by a single measurement if geometry known to be nested



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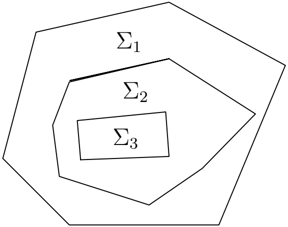
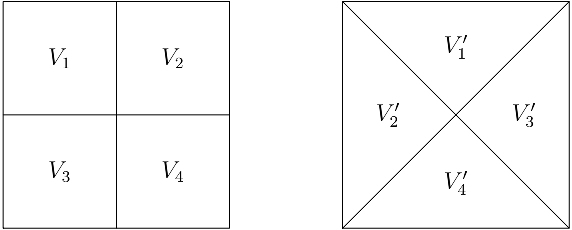


- method cannot yet be shown to distinguish between



Non-scattering

Inverse source problem

$$(\Delta + k^2)u = f, \qquad \lim_{r\to\infty} (\partial_r - ikr)\, u = 0$$

can one have $f \neq 0$ but $u_\infty = 0$?

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Recall:

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Yes: let

$$f(x) = \begin{cases} 1, & |x| < r_0 \\ 0, & |x| \geq r_0 \end{cases}$$

where $r_0 > 0$.

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Yes: let

$$f(x) = \begin{cases} 1, & |x| < r_0 \\ 0, & |x| \geq r_0 \end{cases}$$

where $r_0 > 0$. Then

$$u_\infty(\theta) = c_{k,n}\hat{f}(k\theta) = c'_{k,n}J_{n/2}(kr_0) = 0$$

if kr_0 is a zero of the Bessel function of order $n/2$.

Always scattering
Smallness 1/2

A *small* uniform ball always scatters!

Always scattering

Smallness 1/2

A *small* uniform ball always scatters!

Also: any source with small shape always scatters!

Theorem (B., Liu, preprint)

Let $n \geq 2$, $R_m, k \in \mathbb{R}_+$, $0 \leq \alpha \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain of diameter at most R_m and whose complement is connected. Let Ω_c be a component of Ω. The source $f = \chi_\Omega \varphi$ radiates a non-zero far-field pattern at wavenumber k if

$$\big(\operatorname{diam}(\Omega_c)\big)^\alpha \leq C \frac{\sup_{\partial\Omega_c} |\varphi|}{\|\varphi\|_{C^\alpha(\overline{\Omega}_c)}},$$

for some $C = C(k, R_m, n) > 0$.

Always scattering

Smallness 2/2

Proof.

Suppose $(\Delta + k^2)u = \chi_\Omega \varphi$ and $u_\infty = 0$. Then $u_{|\Omega^\complement} = 0$, so $u_{|\Omega_c} \in H_0^2(\Omega_c)$ and $(\Delta + k^2)u = \varphi$ in Ω_c.

Always scattering

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Set $g = \varphi - k^2 u$. Then elliptic regularity implies $g \in C^\alpha(\overline{\Omega}_c)$ with $\|g\|_{C^\alpha} \leq C(n, k, R_m) \|\varphi\|_{C^\alpha}$. Moreover

$$\int_{\Omega_c} g(x) dx = \int_{\Omega_c} 1 \cdot \Delta u dx = 0$$

because $u = \partial_\nu u = 0$ in $\partial\Omega_c$.

Always scattering

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$$\varphi(p)m(\Omega_c) = g(p)m(\Omega_c) = -\int_{\Omega_c} \left(g(x) - g(p)\right) dx$$

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Hence

$$|\varphi(p)|\, m(\Omega_c) \leq \|g\|_{C^\alpha} \int_{\Omega_c} |x - p|^\alpha\, dx \leq \|g\|_{C^\alpha}\, m(\Omega_c)\big(\operatorname{diam}(\Omega_c)\big)^\alpha.$$

Always scattering

High curvature case

Is smallness the true cause for non-scattering?

Always scattering

High curvature case

Is smallness the true cause for non-scattering?

No: high curvature!

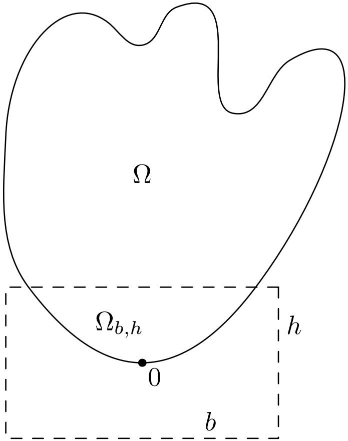
Theorem (B., Liu, preprint)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. Assume that $p \in \partial\Omega$ is an admissible K-curvature point. Also assume p connected to infinity outside of Ω. Consider the source $\chi_\Omega \varphi$, $\varphi \in C^\alpha(\mathbb{R}^n)$. If

$$|\varphi(p)| \geq \mathcal{C}(\ln K)^{(n+3)/2} K^{-\delta}$$

then the source scatters a non-zero far-field pattern at wavenumber k.
Here $\delta > 0$ depends on the geometric parameters, and $\mathcal{C}$ depends on the a-priori parameters of the K-curvature point, the wavenumber k, the upper bound for the diameter of Ω, and the upper bound for $\|\varphi\|_{C^\alpha}$.

Admissible K-curvature point



Inverse source problem, Schiffer's problem

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Unique determination:

- $u_\infty = u'_\infty \implies \Omega = \Omega'$ for convex polyhedral shapes (corner scattering). Also for elasticity (with Lin), electromagnetism (with Liu, Xiao),
- $u_\infty = u'_\infty \implies \Omega \approx \Omega'$ for convex polyhedral shapes whose corners have been smoothened to admissible K-curvature points (high curvature scattering),
- $u_\infty = u'_\infty \implies \Omega \approx \Omega'$ for well-separated collections of small scatterers (small source scattering).

Thank you for your attention!