

Determining the shape of a flat scattering screen with one measurement

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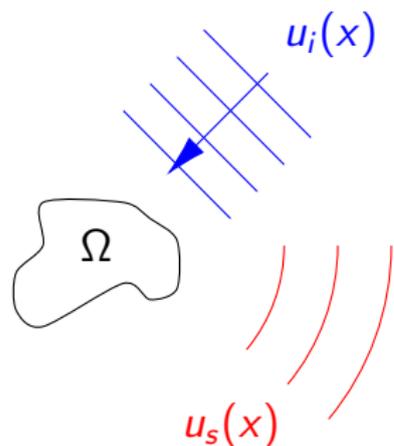
Sections

Shape determination in inverse problems

Flat screen determination

Proofs

Scattering theory



The total wave u satisfies

$$(\Delta + k^2)u = 0, \quad \mathbb{R}^n \setminus \bar{\Omega},$$

& effect on u from the object Ω

The incident wave u_i has

$$(\Delta + k^2)u_i = 0, \quad \mathbb{R}^n.$$

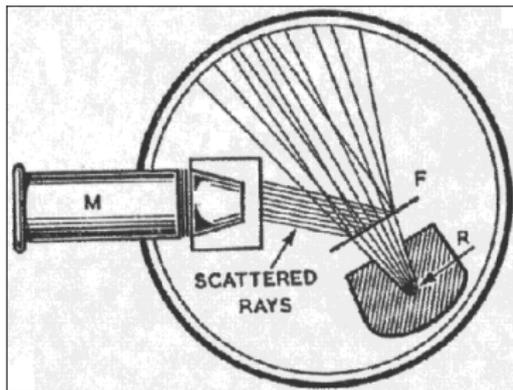
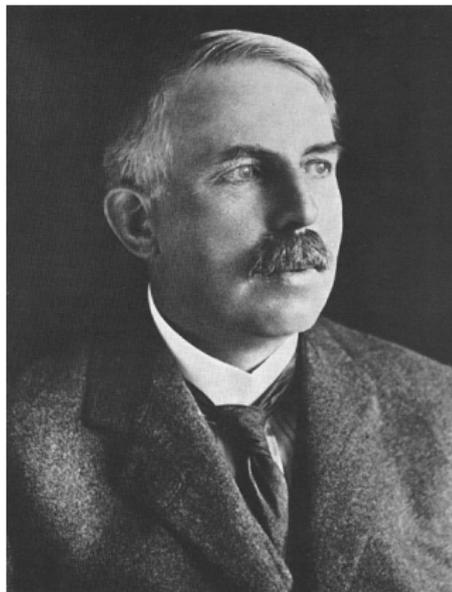
The scattered wave u_s satisfies a **radiation condition at infinity**.

incident wave scattered wave

$$u(x) = u_i(x) + u_s(x)$$

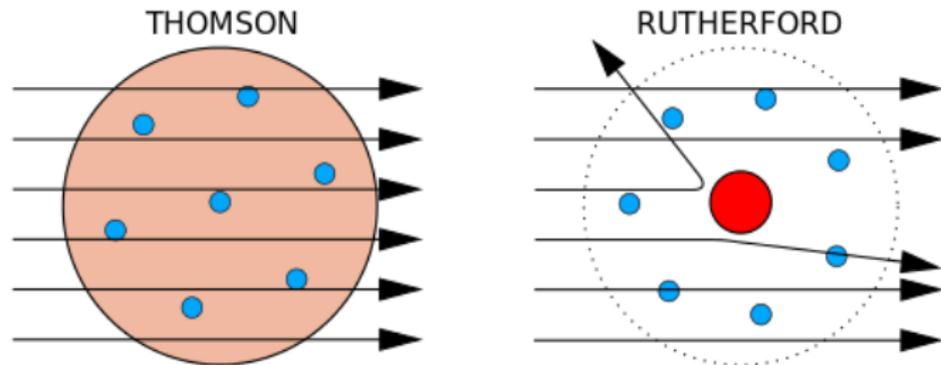
Inverse scattering in real life

Example: Lord Rutherford's gold-foil experiment



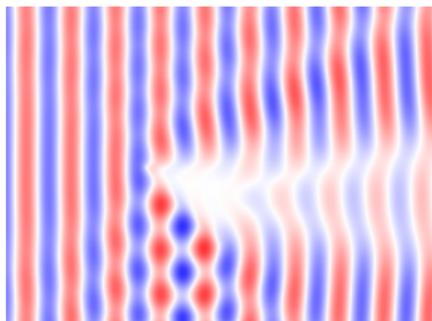
Single incident wave

Rutherford experiment's conclusions

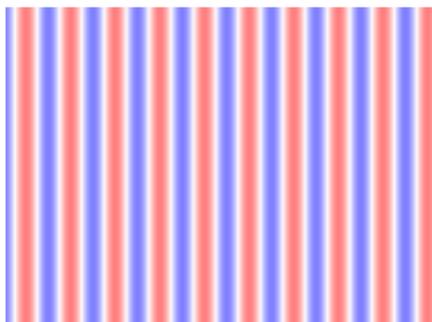


measurement + *a-priori information* = conclusion

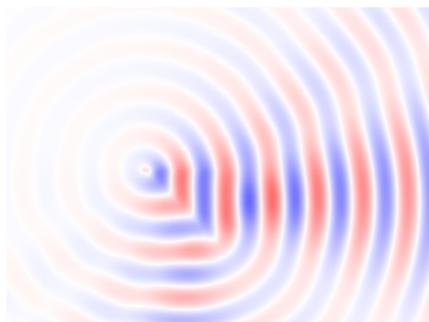
Waves in scattering theory



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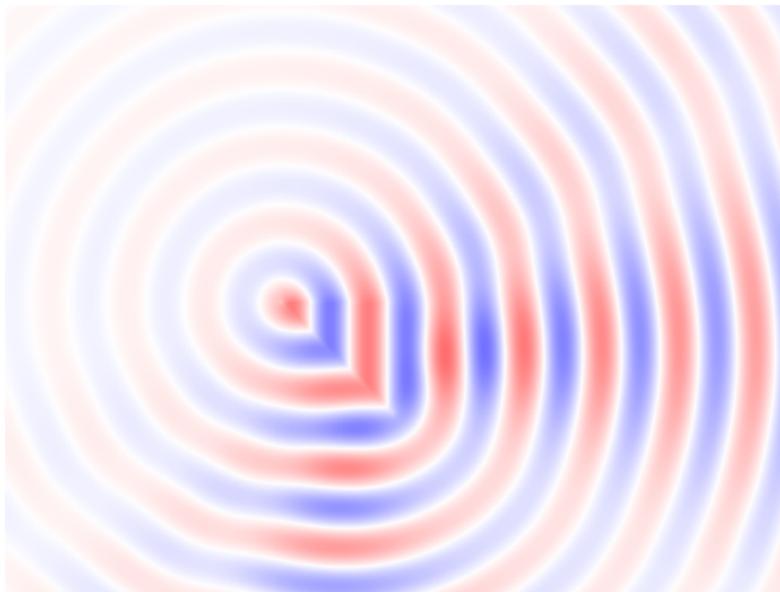


+



$$u = u_i + u_s$$

Inverse scattering: measurements



Measurement: A_{u_i} is the **far-field pattern** of the scattered wave

$$u_s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u_i}(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right), \quad \hat{x} = \frac{x}{|x|}$$

Different inverse scattering problems

Given the *far-field map* $u_j \mapsto A_{u_j}$, recover the scattering potential V , its support Ω or an impenetrable obstacle Ω .

Solved when

- far-field map given for **all waves** of a **single frequency**:
 - Schiffer 1967: Dirichlet obstacle
 - Sylvester–Uhlmann 1987: 3D Calderón problem
 - Nachman, Novikov both in 1988: 3D scattering
 - Bukhgeim 2007: 2D scattering
- + countless variations

My focus is on *single measurement*: A_{u_j} given for **only one** u_j .

Schiffer's problem

Can a single measurement determine Ω ?

Single measurement inverse obstacle scattering

Terminology from acoustics:

- sound-soft $\Leftrightarrow u = 0$ on $\partial\Omega$
- sound-hard $\Leftrightarrow \partial_\nu u = 0$ on $\partial\Omega$

Past results with N different incident waves:

- $N = \infty$ sound-soft obstacle (Schiffer 1987)
- $N < \infty$ obstacle, sound-soft (Colton–Sleeman 1983), sound-hard (Kirsch–Kress 1993)
- Alves–Ha-Duong 1997: $N = \infty$ for scattering surface, $N = 1$ **plane wave** for sound-soft flat screen
- Admissible / polyhedral obstacles and screens: Rondi 2003 $N < \infty$, Cheng–Yamamoto 2003 $N = 1, 2$, Alessandrini–Rondi 2005 sound-soft $N = 1$, Liu–Zou 2006 sound-hard $N = n$, Rondi 2008 and Liu–Petrini–Rondi–Xiao 2016 stability, ...

Single measurement vs infinitely many

With one measure, you only have one solution to your PDE to work with!

⇒ you need a deeper understanding of the direct problem!

⇒ you need to analyse how solutions behave near boundaries or other points of interest!

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Scattering from a flat screen

Definition

$\Omega \subset \mathbb{R}^3$ is a **flat screen** of $\Omega = \Omega_0 \times \{0\}$ for some simply connected bounded domain $\Omega_0 \subset \mathbb{R}^2$ whose boundary is smooth.

Definition

Let $u_i: \mathbb{R}^3 \rightarrow \mathbb{C}$ satisfy $(\Delta + k^2)u_i = 0$ in \mathbb{R}^3 and let Ω be a flat screen. $u = u_i + u_s \in H_{loc}^1(\mathbb{R}^3 \setminus \overline{\Omega})$ solves the **direct screen scattering problem** if

$$\begin{aligned}(\Delta + k^2)u_s &= 0, & \mathbb{R}^3 \setminus \overline{\Omega}, \\ u_i(x) + u_s(x) &= 0, & x \in \Omega, \\ r(\partial_r - ik)u_s &= 0, & r \rightarrow \infty,\end{aligned}$$

where $r = |x|$ and the limit is uniform in all directions $\hat{x} \in \mathbb{S}^2$.

Inverse problem statement

Definition

Let u_s satisfy the Sommerfeld radiation condition and $(\Delta + k^2)u_s = 0$ outside a ball $B \subset \mathbb{R}^3$. We say $u_s^\infty : \mathbb{S}^2 \rightarrow \mathbb{C}$ is the **far-field pattern** of u_s , if

$$u_s(x) = \frac{e^{ik|x|}}{|x|} \left(u_s^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right)$$

uniformly over \hat{x} as $|x| \rightarrow \infty$.

Inverse problem

Given $u_s^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}^2$ and one $k > 0$, can we determine the shape Ω_0 of a flat screen $\Omega = \Omega_0 \times \{0\}$?

Our results (Blåsten, Päivärinta, Sadique 2020)

Theorem

$\Omega \subset \mathbb{R}^3$ a flat screen and u_s satisfies the direct problem. Then

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \{0\}} e^{-ik\hat{x} \cdot y} (\partial_3 u_s^+ - \partial_3 u_s^-)(y) dy$$

for $\hat{x} \in \mathbb{S}^2$. f^\pm are the one-sided limits to $\mathbb{R}^2 \times \{0\}$.

Theorem

$\Omega, \tilde{\Omega}$ flat screens, $k > 0$, u_i an incident wave. Let u_s, \tilde{u}_s satisfy the direct problem for $\Omega, \tilde{\Omega}$, respectively. Assume $u_s^\infty = \tilde{u}_s^\infty$.

- If $u_i(x_1, x_2, x_3) \neq -u_i(x_1, x_2, -x_3)$ at some point, then $\Omega = \tilde{\Omega}$.
- If $u_i(x_1, x_2, x_3) = -u_i(x_1, x_2, -x_3)$ everywhere, then $u_s^\infty = \tilde{u}_s^\infty = 0$ for all $\Omega, \tilde{\Omega}$.

Comparison to past result

Theorem (Alves, Ha-Duong 1997: by plane waves)

Consider the screens Ω_1, Ω_2 in the plane $\{x_3 = 0\}$ where we have Dirichlet boundary conditions and only one incident *plane wave* with direction d . If the associated far fields F_1, F_2 (of the scattered waves) are equal, i.e.

$$F_1(\hat{x}) = F_2(\hat{x}) \quad \forall \hat{x} \in \mathbb{S}^2$$

we have $\Omega_1 = \Omega_2$.

In our result, we can have *any* incident wave. Even “bad” ones that vanish on Ω .

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Far-field representation, part 1/2

$$\Phi(x, y) = \frac{\exp(ik|x - y|)}{4\pi|x - y|}, \quad (\Delta_x + k^2)\Phi = \delta_y(x)$$

For $x \in D$ with $\Omega \subset \partial D$, $D \subset \mathbb{R}^2 \times \mathbb{R}_+$:

$$\begin{aligned} u_s(x) &= - \int_D \Phi(x, y)(\Delta + k^2)u_s(y) dy \\ &\quad + \int_{\partial D} (\Phi(x, y)\partial_\nu u_s(y) - u_s(y)\partial_\nu \Phi(x, y)) ds(y) \\ 0 &= - \int_{B \setminus \bar{D}} \Phi(x, y)(\Delta + k^2)u_s(y) dy \\ &\quad + \int_{\partial(B \setminus \bar{D})} (\Phi(x, y)\partial_\nu u_s(y) - u_s(y)\partial_\nu \Phi(x, y)) ds(y) \end{aligned}$$

Split $\partial(B \setminus \bar{D})$, let radius of B grow, use Sommerfeld radiation...

$$u_s(x) = \int_{\mathbb{R}^2 \times \{0\}} \Phi(x, y)(\partial_3 u_s^+ - \partial_3 u_s^-)(y) dy$$

Far-field representation, part 2/2

For y in a compact set K and $|\alpha| \leq 1$

$$\lim_{r \rightarrow \infty} \sup_{|x|=r} \sup_{y \in K} |x| \left| \partial_y^\alpha \left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x} \cdot y} \right) \right| = 0$$

Definition of u_s^∞ , namely $u_s(x) = e^{ik|x|} |x|^{-1} (u_s^\infty(\hat{x}) + \mathcal{O}(|x|^{-1}))$, gives

$$\begin{aligned} u_s^\infty(\hat{x}) &= \lim_{|x| \rightarrow \infty} \frac{|x|}{e^{ik|x|}} u_s(|x|\hat{x}) \\ &= \lim_{|x| \rightarrow \infty} \frac{|x|}{e^{ik|x|}} \int_{\mathbb{R}^2 \times \{0\}} \frac{\exp(ik|x-y|)}{4\pi|x-y|} (\partial_3 u_s^+ - \partial_3 u_s^-)(y) dy \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \{0\}} e^{-ik\hat{x} \cdot y} (\partial_3 u_s^+ - \partial_3 u_s^-)(y) dy, \end{aligned}$$

which is our first theorem. It was ok since $\partial_3 u_s^+ - \partial_3 u_s^- \in H^{-1/2}$, and the difference of the exponentials had $|\alpha| \leq 1$ above. ■

Solving the inverse problem, part 1

Setting

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \{0\}} e^{-ik\hat{x} \cdot y} \rho(y_1, y_2) dy, \quad \hat{x} \in \mathbb{S}^2$$

implies that $\rho = \rho(x_1, x_2)$ can be determined from u_s^∞ :

$$u_s^\infty(\hat{x}) = \frac{1}{2} \mathcal{F} \rho(k\hat{x}_1, k\hat{x}_2)$$

so

$$\mathcal{F} \rho(\xi_1, \xi_2) = 2u_s^\infty(\xi_1/k, \xi_2/k, \sqrt{1 - (\xi_1/k)^2 - (\xi_2/k)^2})$$

and then invert the Fourier transform.

This gives us $\partial_3 u_s^+ - \partial_3 u_s^-$ from u_s^∞ . We need to then determine Ω .

Solving the inverse problem, part 2

Define

$$\rho(x_1, x_2) = (\partial_3 u_s^+ - \partial_3 u_s^-)(x_1, x_2, 0).$$

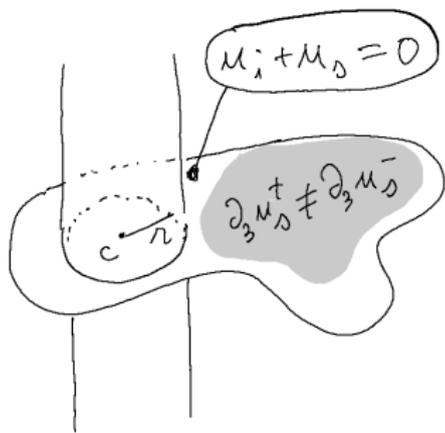
It is a $H^{-1/2}(\Omega_0)$ -distribution. We will show that $\overline{\Omega}_0 = \text{supp } \rho$.

- $\text{supp } \rho \subset \overline{\Omega}_0$ because of elliptic regularity of u_s outside of Ω .
- If $\overline{\Omega}_0 \not\subset \text{supp } \rho$ then $\exists c, r$ such that $B(c, r) \subset \Omega_0 \setminus \text{supp } \rho$.
Let's investigate the behaviour of u_s and u_i in the tube $B(c, r) \times \mathbb{R}$ (on the whiteboard).

... ..
Then $u_i(x_1, x_2, x_3) = -u_i(x_1, x_2, -x_3)$ for all $x \in \mathbb{R}^3$. ■

$$\textcircled{1} \mu_D(x) = \int_{\mathbb{R}^2 \times \{0\}} \Phi(x,y) f(y) dy$$

$$\Rightarrow \mu_D(x_{11}, x_{21}, x_3) = \mu_D(x_{11}, x_{21}, -x_3).$$



$$\textcircled{2} \partial_3 \mu_D^+ = \partial_3 \mu_D^- \text{ in tube \& } \textcircled{1}$$

$$\Rightarrow \partial_3 \mu_D(x_{11}, x_{21}, 0) = 0$$

for $(x_{11}, x_{21}) \in B(c, r)$.

\textcircled{3} From \textcircled{1} & \textcircled{2} & definition of Ω :

$$\begin{cases} \mu_D = -\mu_i \\ \partial_3 \mu_D = 0 \end{cases}$$

on $B(c, r) \times \{0\}$.

\textcircled{4} Higher derivatives in the tube

$$\begin{aligned} 0 &= \partial_3^m (\Delta + k^2) \mu_D = (\Delta + k^2) \partial_3^m \mu_D \\ &= (\partial_1^2 + \partial_2^2 + k^2) \partial_3^m \mu_D + \partial_3^{m+2} \mu_D \end{aligned}$$

\textcircled{5} Hence \textcircled{3} & \textcircled{4} give

$$\partial_3^j \mu_D = \begin{cases} (-1)^{j/2} (\Delta + k^2)^{j/2} \mu_i, & j \text{ even} \\ 0, & j \text{ odd} \end{cases}$$

on $B(c, r) \times \{0\}$.

⑥ u_i is a wave, $(\Delta + k^2)u_i = 0$.

$$\Rightarrow (\partial_1^2 + \partial_2^2 + \partial_3^2)u_i = -\partial_3^2 u_i$$

$$\Rightarrow (\partial_1^2 + \partial_2^2 + \partial_3^2)^j u_i = (-\partial_3^2)^j u_i$$

⑦ By ⑤ & ⑥, on $B(c, r) \times \{0\}$.

$$\partial^\alpha u_\Delta = \begin{cases} -\partial^\alpha u_i & , \alpha_3 \text{ even} \\ 0 & , \alpha_3 \text{ odd} \end{cases}$$

⑧ Define

$$\tilde{u}_i(x) = \frac{1}{2}(u_i(x_1, x_2, x_3) + u_i(x_1, x_2, -x_3))$$

$$\text{then } \partial^\alpha \tilde{u}_i = \begin{cases} \partial^\alpha u_i & , \alpha_3 \text{ even} \\ 0 & , \alpha_3 \text{ odd} \end{cases}$$

so

$$\partial^\alpha u_\Delta = -\partial^\alpha \tilde{u}_i \quad \forall \alpha$$

on $B(c, r) \times \{0\}$.

⑨ Real-analyticity of u_Δ, \tilde{u}_i in the tube and away from Ω and ⑧

$$\Rightarrow u_\Delta = -\tilde{u}_i \text{ in } \mathbb{R}^3 \setminus \bar{\Omega}.$$

⑩ u_Δ satisfies the Sommerfeld radiation condition $\Rightarrow \tilde{u}_i$ also.

$$\text{But } (\Delta + k^2)\tilde{u}_i(x) = 0 \quad \forall x \in \mathbb{R}^3.$$

$$\Rightarrow \tilde{u}_i(x) = 0 \quad \forall x \in \mathbb{R}^3.$$

Hence

$$u_i(x_1, x_2, x_3) = -u_i(x_1, x_2, -x_3) \\ \text{for all } x \in \mathbb{R}^3. \quad \square$$