

# Imaging water supply pipes using pressure waves

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group

COPE-INAR workshop

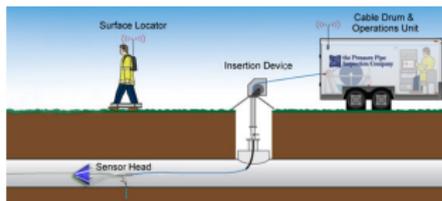
LUT University,  
November 17, 2022

# Water supply network



# How to locate problems traditionally?

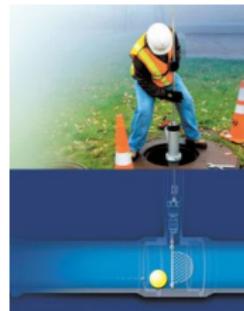
## Sahara System



## Replace & Rehabilitate



## Smart Ball



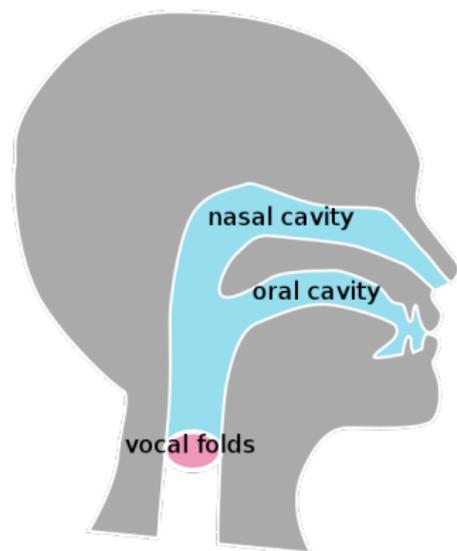
## Sonar



## Gas Injection



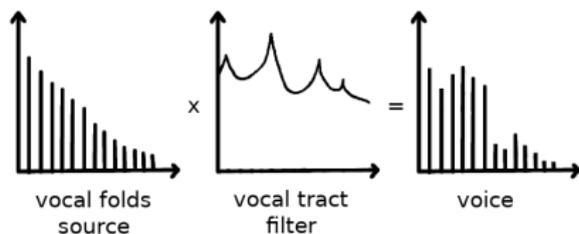
# Vocal tract imaging: inverse glottal filtering



● source

● filter

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Wikipedia user Emflazie



output = source \* filter

Find cheap and convenient ways to get *useful information* about the source or the filter!

## Some examples of useful information

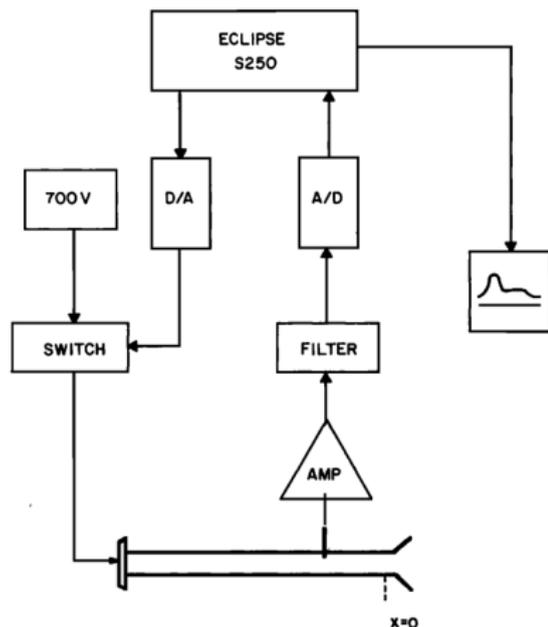
In voice therapy:

- ▶ vocal tract length
- ▶ vocal tract volume
- ▶ resonant frequencies of vocal tract
- ▶ open/closed quotient
- ▶ source spectral slope
- ▶ ...

Difficulty: IGF is a *blind deconvolution*!

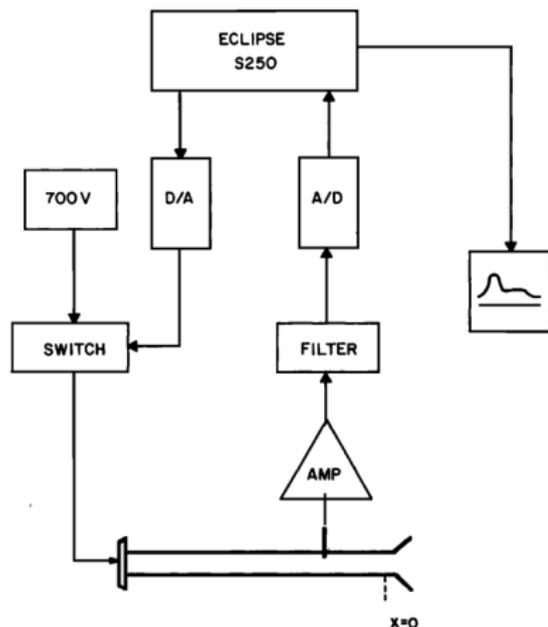
However: the above are all single numbers! Some rudimentary methods exist already, but not considered in this context.

## Vocal tract imaging: with an external sound



Sondhi, M. M., & Resnick, J. R. (1983). The inverse problem for the vocal tract: numerical methods, acoustical experiments, and speech synthesis. *The Journal of the Acoustical Society of America*, 73(3), 985–1002. <http://dx.doi.org/10.1121/1.389024>

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Alternative source: white noise, it contains all the frequencies!

# Mathematical model

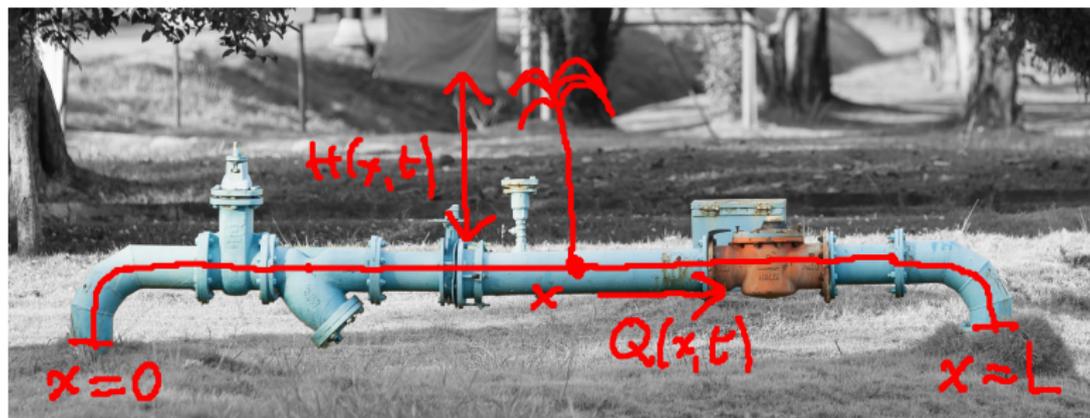
Water in pipes  
VS  
Air in the vocal tract

Fluid in a pipe. Same model!

## Direct model: single pipe



## Direct model: single pipe



$$\begin{aligned} \partial_t H + \frac{a^2}{gA} \partial Q &= 0, & 0 < x < L, & \quad t \in \mathbb{R}, \\ \partial_t Q + gA \partial H &= 0, & 0 < x < L, & \quad t \in \mathbb{R}, \\ H = Q &= 0, & 0 < x < L, & \quad t \leq 0. \end{aligned}$$

Water hammer equations.

<https://www.youtube.com/watch?v=jTrhHUwDNYE>

## One pipe inverse problem

Measurement  $\Lambda(t)$  defined by assuming  $H, Q$  satisfy

$$\begin{aligned}\partial_t H + \frac{a^2}{gA} \partial Q &= 0, & 0 < x < L, & \quad t \in \mathbb{R}, \\ \partial_t Q + gA \partial H &= 0, & 0 < x < L, & \quad t \in \mathbb{R}, \\ H = Q &= 0, & 0 < x < L, & \quad t \leq 0\end{aligned}$$

Boundary conditions:

unknown at  $x = L$

unit impulse discharge  $Q(0, t) = \delta_0(t)$  at  $x = 0$

Then  $\Lambda(t) = H(0, t)$ .

Recover:  $A(x)$  (or  $a(x)$ , or  $A(x)/a(x)$ )

## Integration by parts

For simplicity assume  $a(x) = a_0$  constant! Assume virtual  $H_v, Q_v$  causal solutions. Then

$$-\partial Q_v = \frac{gA}{a_0^2} \partial_t H_v$$

and integrate  $\int_0^\tau \int_0^{a_0\tau} \dots dxdt$  given any fixed  $\tau > 0$ .

$$-\int_0^\tau \int_0^{a_0\tau} \partial Q_v(x, t) dxdt = \int_0^\tau \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} \partial_t H_v(x, t) dxdt$$

- ▶  $H_v = Q_v = 0$  at  $t = 0$
- ▶ hence  $H_v(x, t) = Q_v(x, t) = 0$  when  $x \geq a_0 t$ , so

$$\int_0^\tau Q_v(0, t) dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H_v(x, \tau) dx \quad (1)$$

## Special solutions

Given any causal solutions, for example the virtual ones  $H_v, Q_v$ , let's look at the total volume input into the system:

$$V(\tau) := \int_0^\tau Q_v(0, t) dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H_v(x, \tau) dx.$$

If  $H_v$  is such that

$$H_v(x, \tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geq a_0\tau \end{cases} \quad \text{at } t = \tau,$$

then

$$A(x) = \frac{a_0}{g} \frac{\partial V}{\partial \tau} \left( \frac{x}{a_0} \right)$$

## After the facts, new problem statement

Unknown:  $A(x)$ . Measurement:

$$Q(0, t) = \delta_0(t), \quad H(0, t) = \Lambda(t).$$

Given any  $\tau$ , find causal solutions  $H_v, Q_v$  such that at time  $t = \tau$

$$H_v(x, \tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geq a_0\tau \end{cases}$$

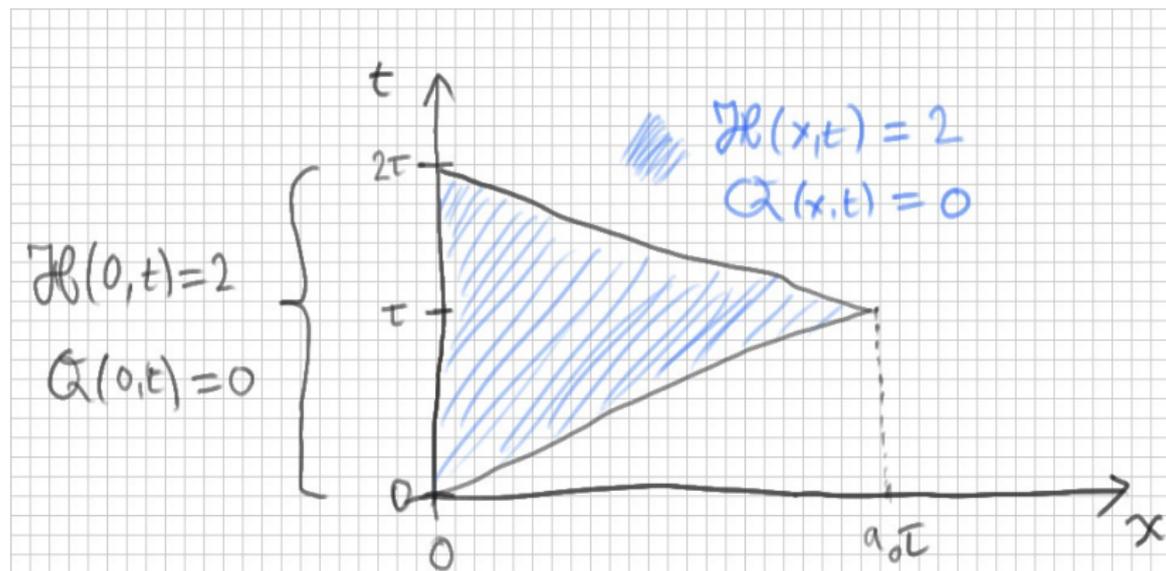
The corresponding  $Q_v$  will be used to calculate  $A(x)$ .

## Unique continuation

If  $\mathcal{H}, \mathcal{Q}$  satisfy the equations on  $0 < x < L, 0 < t < 2\tau$  (but are not necessarily causal), and

$$\mathcal{H}(0, t) = 2, \quad \mathcal{Q}(0, t) = 0, \quad 0 < t < 2\tau,$$

then  $\mathcal{H}(x, t) = 2, \mathcal{Q}(x, t) = 0$  in  $x < a_0(\tau - |\tau - t|)$ .



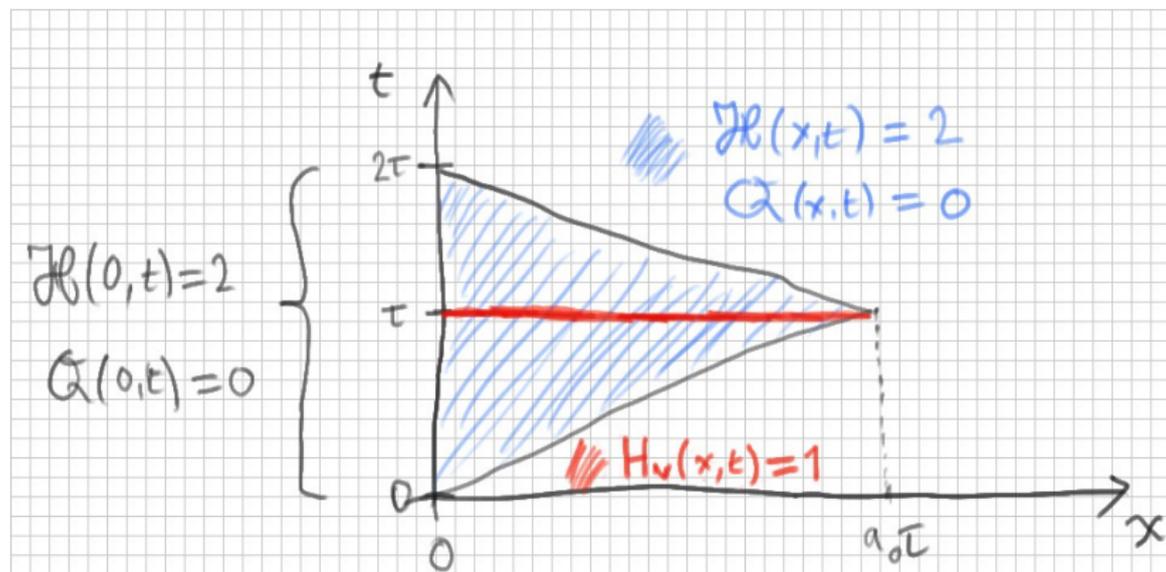
## Unique continuation to causal solutions

If  $\mathcal{H}$ ,  $\mathcal{Q}$  as previously and  $H_v, Q_v$  causal, then

$$\mathcal{H}(x, t) = H_v(x, t) + H_v(x, 2\tau - t), \quad \mathcal{Q}(x, t) = Q_v(x, t) - Q_v(x, 2\tau - t)$$

$$\implies H_v(x, \tau) = \frac{1}{2} \mathcal{H}(x, \tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geq a_0\tau \end{cases}$$

Then  $A(a_0\tau) = a_0 g^{-1} \partial_\tau \int_0^\tau Q_v(0, t) dt$ . (Note:  $Q_v$  depends on  $\tau$ !!)



Next?

How to find the suitable  $H_v, Q_v$ ?

## Integral equation from requirements of $\mathcal{H}$ , $\mathcal{Q}$

Measurement:

$$Q(0, t) = \delta_0(t), \quad H(0, t) = \frac{a_0}{gA(0)}(\delta_0(t) + h(t))$$

Let  $H_v, Q_v$  be causal solutions such that  $\mathcal{H}(0, t) = 2$ ,  $\mathcal{Q}(0, t) = 0$  on  $0 < t < 2\tau$ . Then rewriting these eq's in term of  $Q_v$  gives:

$$Q_v(0, t) + \frac{1}{2} \int_0^{2\tau} Q_v(0, s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \quad 0 < t < 2\tau.$$

Conversely, if  $Q_v$  solves the above and  $H_v$  is the corresponding pressure head, then

$$H_v(x, \tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geq a_0\tau \end{cases} \quad \text{at } t = \tau,$$

so  $A(a_0\tau) = a_0g^{-1}\partial_\tau \int_0^\tau Q_v(0, t)dt$

## Algorithm

1. Input  $Q(0, t) = \delta_0(t)$  and for  $t < 2T = 2L/a_0$  measure

$$H(0, t) = \frac{a_0}{gA(0)}(\delta_0(t) + h(t))$$

2. For  $0 < \tau < T$  solve for the boundary value of the virtual solution  $Q_v$

$$Q_v(0, t) + \frac{1}{2} \int_0^{2\tau} Q_v(0, s) h(|s - t|) ds = \frac{gA(0)}{a_0}, \quad 0 < t < 2\tau$$

3. Set

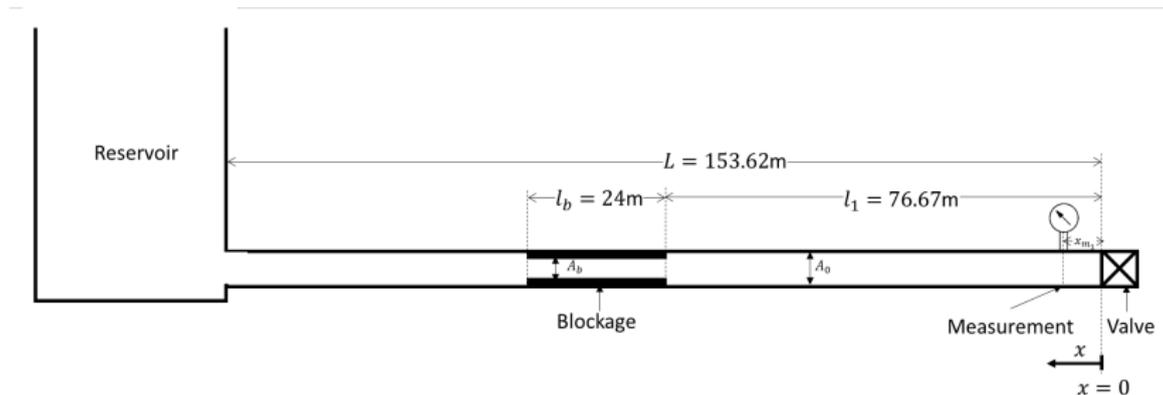
$$V(\tau) = \int_0^\tau Q_v(0, t) dt \quad \left( = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} dx \right)$$

4. Repeat 2–3 (on the computer) for many  $\tau$  to get a good approximation of  $V$
5. Given  $x < L$  the area can be found by

$$A(x) = \frac{a_0}{g} \left( \frac{\partial}{\partial \tau} V(\tau) \right)_{\tau=x/a_0}$$

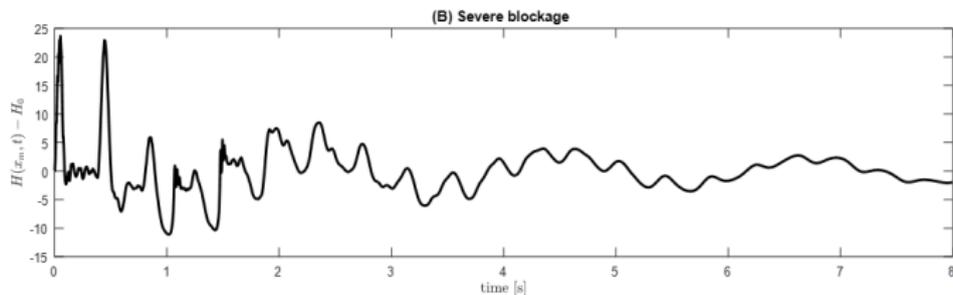
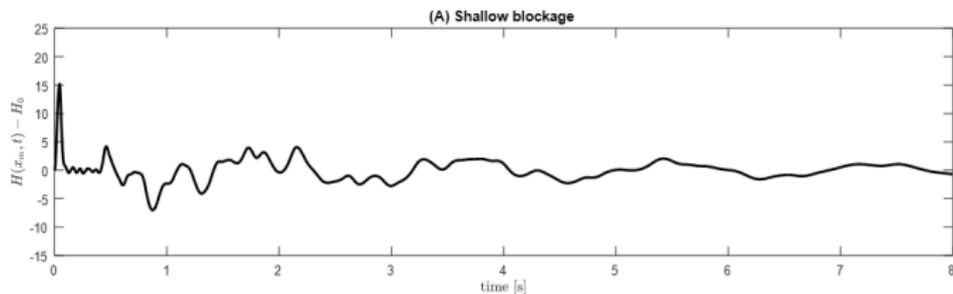
## Laboratory experiment: setup

Measurement set up by Silvia Meniconi and Bruno Brunone's group.



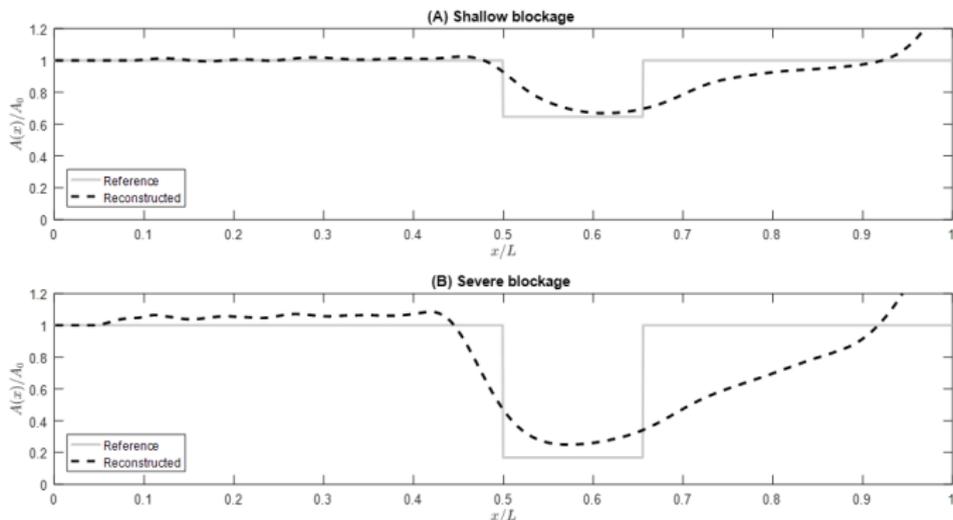
# Laboratory experiment: impulse-response function

Measurement set up by Silvia Meniconi and Bruno Brunone's group.

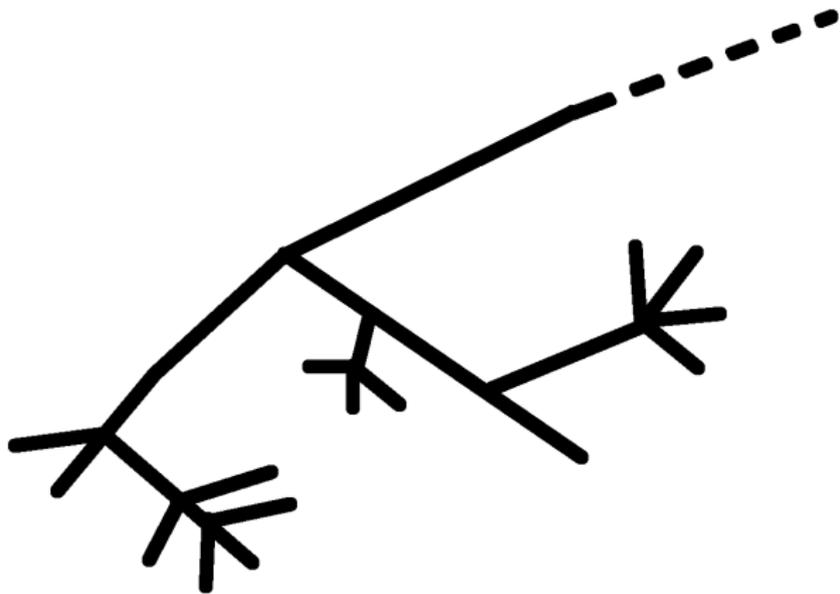


# Reconstruction from measured and processed data

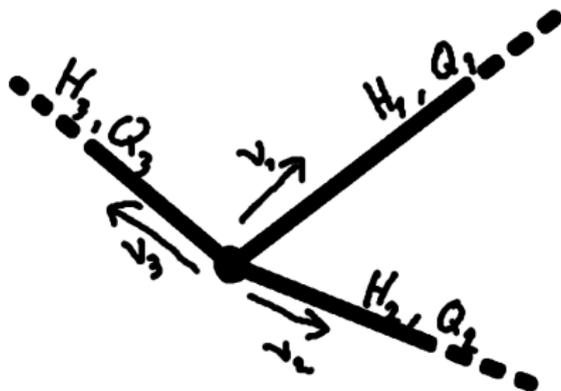
Measurement set up by Silvia Meniconi and Bruno Brunone's group.



# Network



## Junction conditions



- ▶  $H$  is a scalar: the boundary values  $H_j$  are the same at connected pipe ends
- ▶ No sinks or sources (total water flowing into pipes sum to zero):

$$\sum_j \nu_j Q_j = 0$$

## Main difficulty compared to a segment?

The sets where we can have  $H_V(x, \tau) = 1$ .

In which  $\Omega$  can we force  $H_v(x, \tau) = 1$ ?

Control theory suggests that there are boundary values such that

$$H_v(x, \tau) = \begin{cases} 1, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

given any measurable set  $\Omega \subset \mathbb{G}$  when  $\mathbb{G}$  is a tree and  $\tau$  large.

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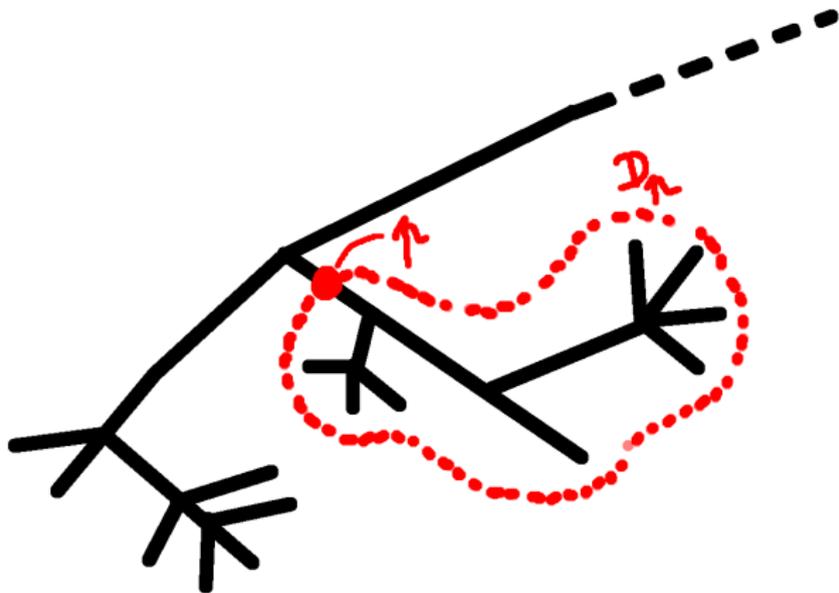
given any measurable set  $\Omega \subset \mathbb{G}$  when  $\mathbb{G}$  is a tree and  $\tau$  large.

## HOWEVER

Can we solve for these boundary values? Is it computationally efficient? Is it even possible?

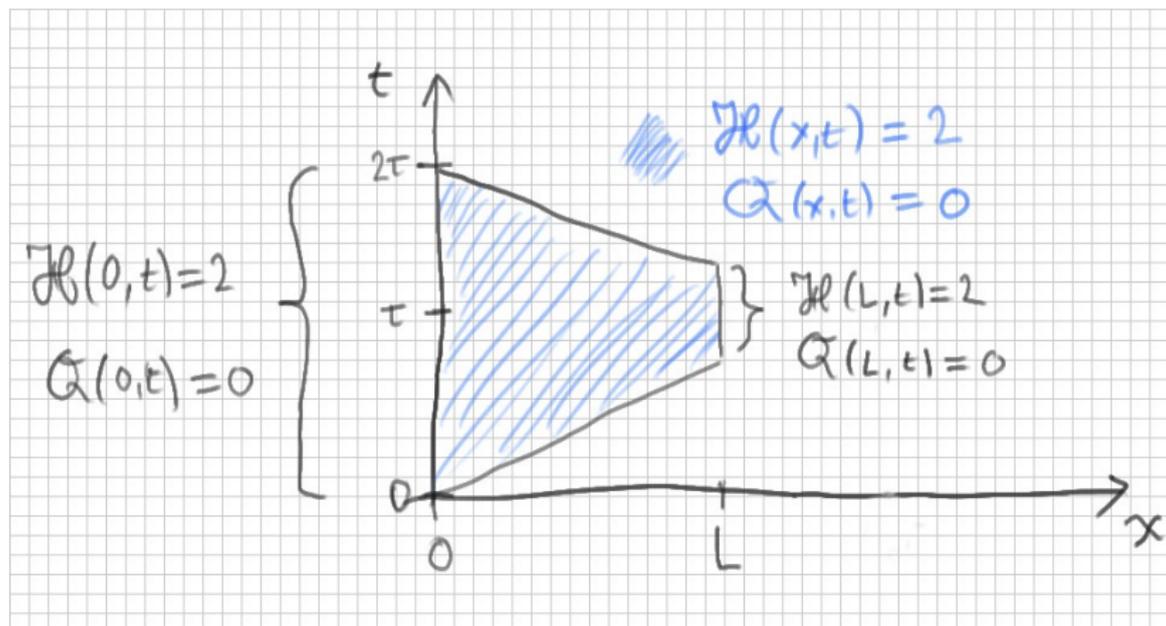
## Admissible domains

This works:



But needs a matrix of measurements!

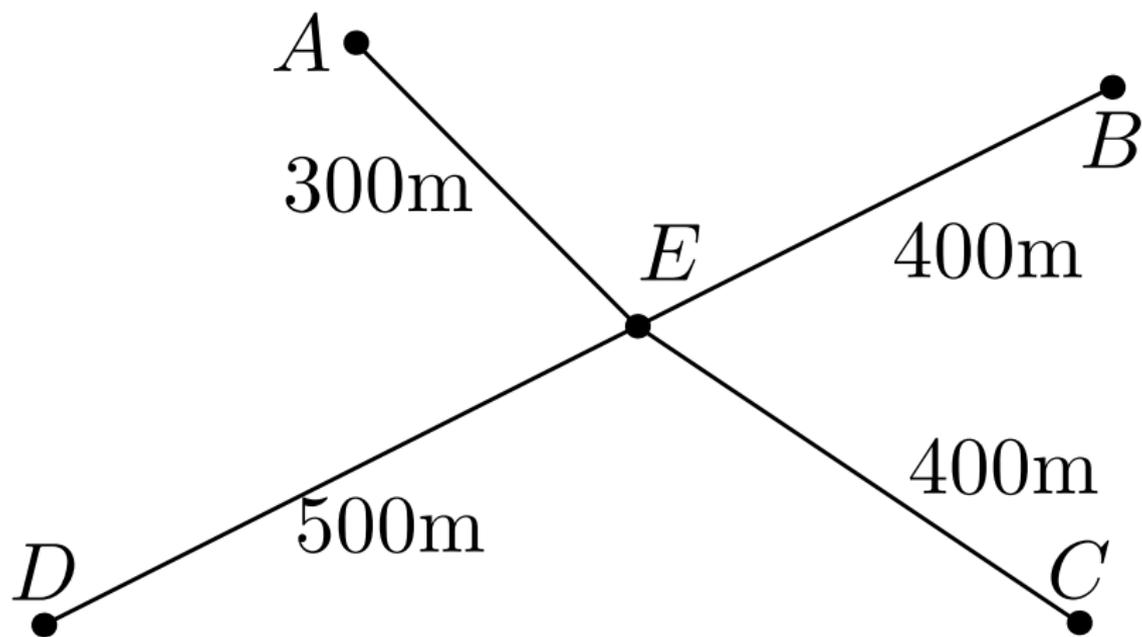
## Inductive unique continuation



+ junction conditions  
= propagate  $\mathcal{H} = 2$ ,  $\mathcal{Q} = 0$ !!

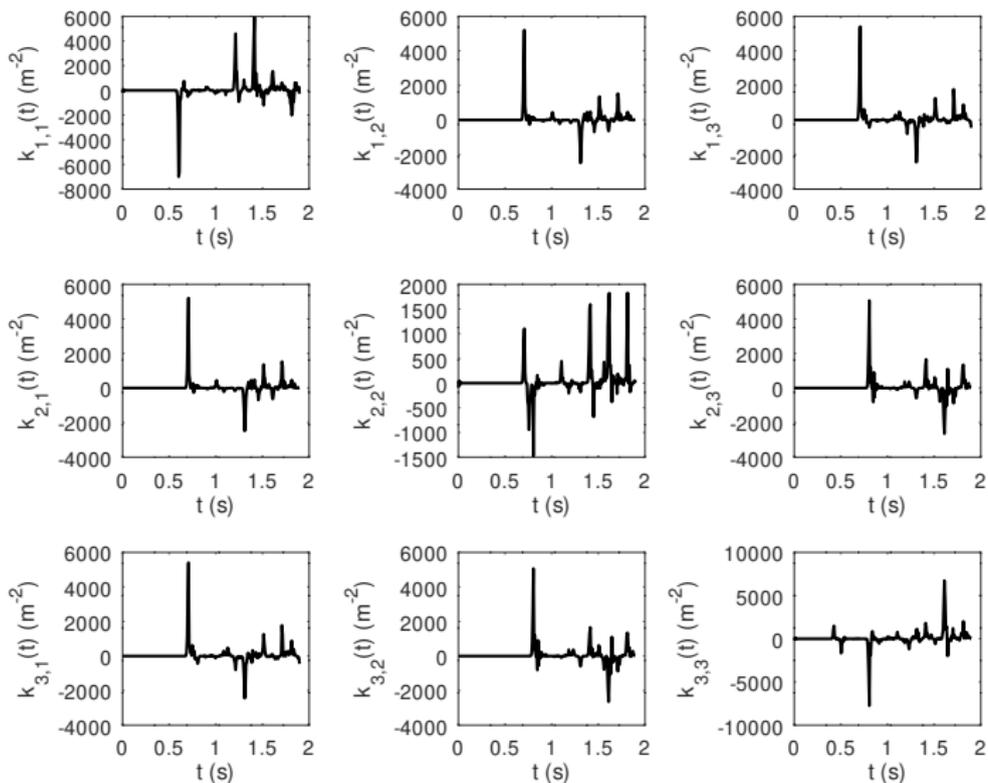
The same logic as before works, but all the equations become more complicated.

## Numerical experiment: setup

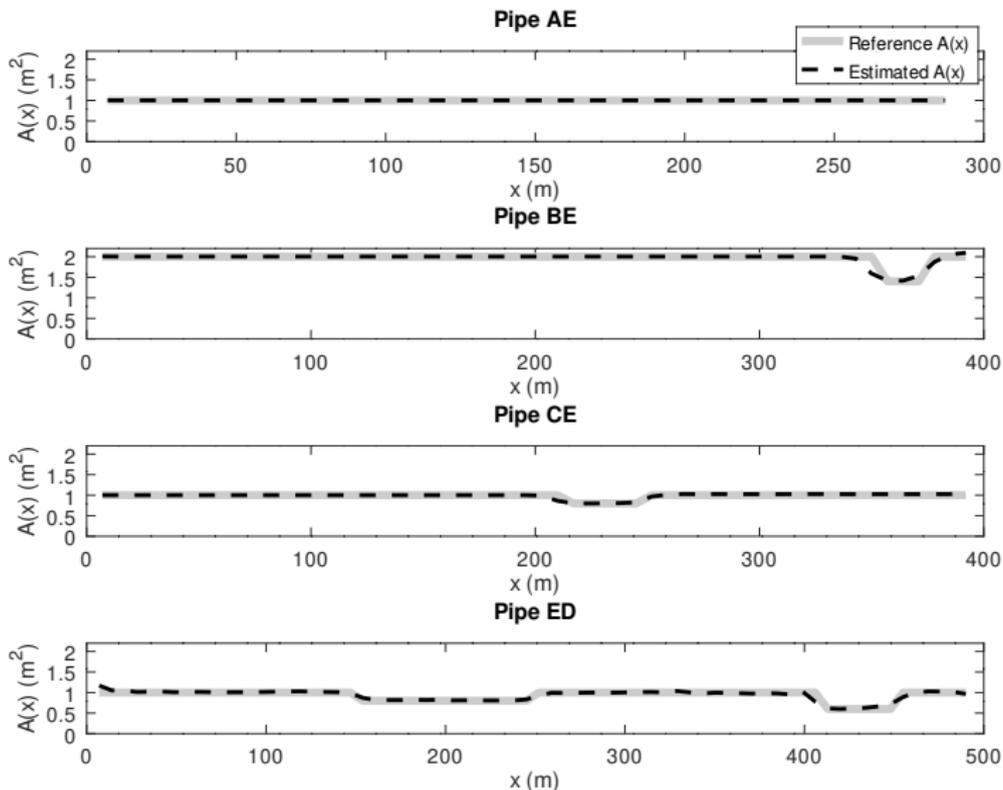


# Numerical experiment: impulse-response matrix function

$$h_{ij}(t) = A(x_j)g/a_0 \cdot k_{ij}(t)$$



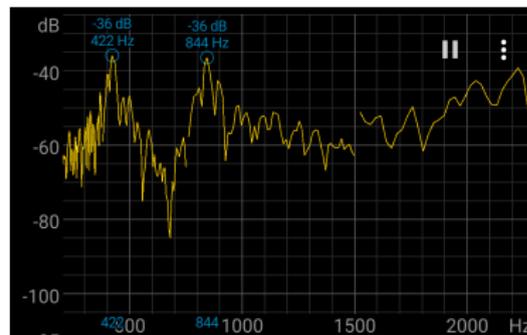
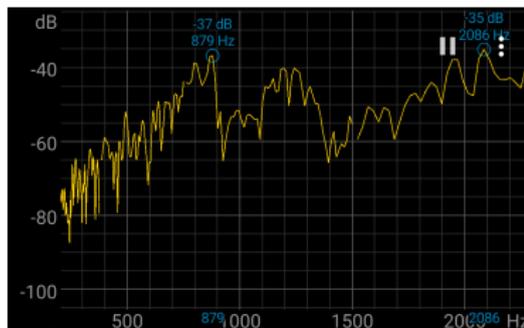
# Reconstruction from measured data using regularization



# Ongoing work

With Lauri Oksanen (Univ. Helsinki) and Tapio Helin (LUT)

Instead of  $Q(0, t) = \delta_0(t)$  we have white noise  $Q(0, t) = W(t)$ .



Kiitos!