

# Corners always scatter — quantitative results

Eemeli Blåsten

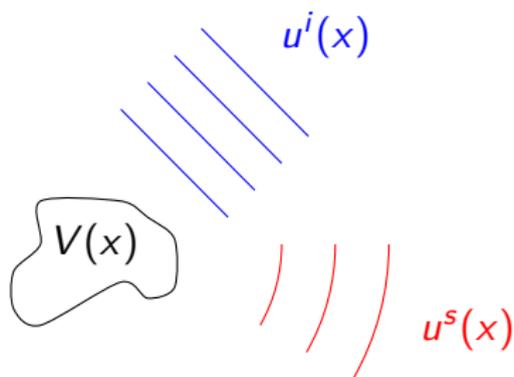
Institute for Advanced Study,  
The Hong Kong University of Science and Technology

Applied Inverse Problems 2017

Hangzhou, May 31

# Scattering theory

Fixed frequency scattering



The total wave  $u$  satisfies

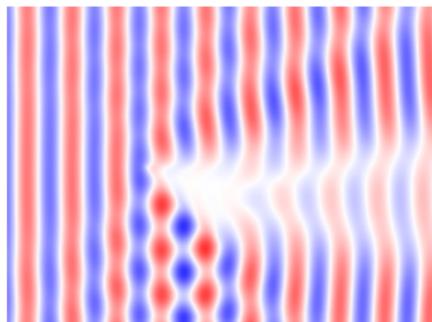
$$(\Delta + k^2(1 + V))u = 0,$$

$V$  models a **perturbation** of the background,

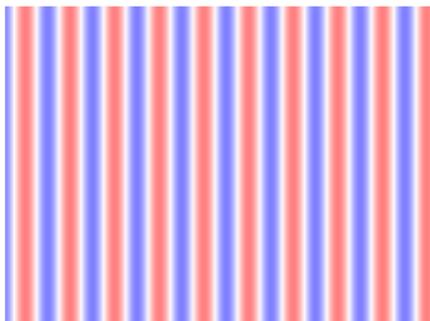
$$u = u^i(x) + u^s(x)$$

$\uparrow$  incident wave       $\nwarrow$  scattered wave

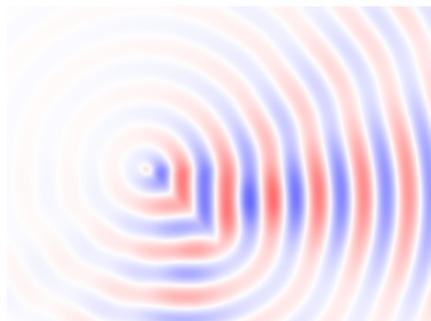
# Scattering theory



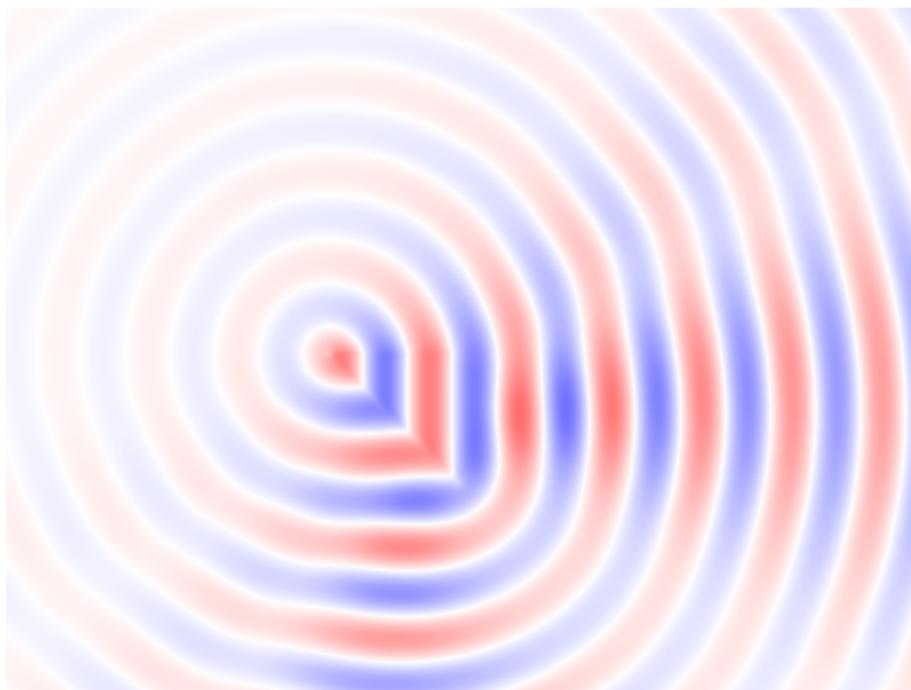
=



+



## Mathematical scattering theory: measurements



Measurement:  $A_{u^i}$  is the **far-field pattern** of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i} \left( \frac{x}{|x|} \right) + \mathcal{O} \left( \frac{1}{|x|^{n/2}} \right)$$

## Inverse problem

Given the map  $u^j \mapsto A_{u^j}$ , recover  $V$  or its support  $\Omega$ .

Early methods (< 85')

- ▶ optimization and minimization methods

## Inverse problem

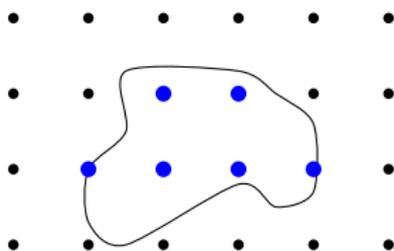
Given the map  $u^j \mapsto A_{u^j}$ , recover  $V$  or its support  $\Omega$ .

Early methods (< 85')

- ▶ optimization and minimization methods

Sampling methods

- ▶ looks for  $\text{supp } V$
- ▶ compared to before: *fast! works reliably!*



## Inverse problem

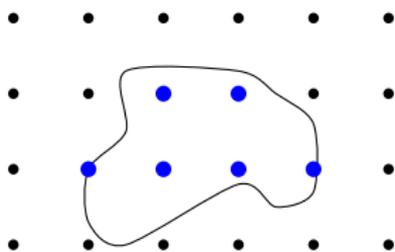
Given the map  $u^j \mapsto A_{u^j}$ , recover  $V$  or its support  $\Omega$ .

Early methods (< 85')

- ▶ optimization and minimization methods

Sampling methods

- ▶ looks for supp  $V$
- ▶ compared to before: *fast! works reliably!*

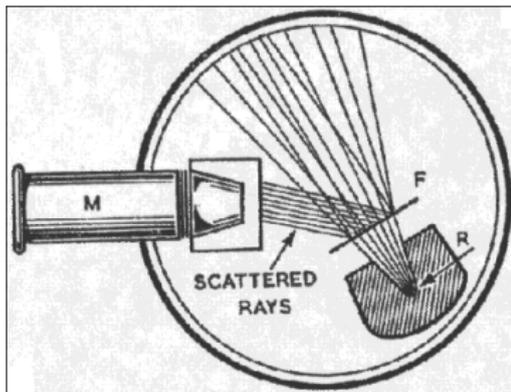
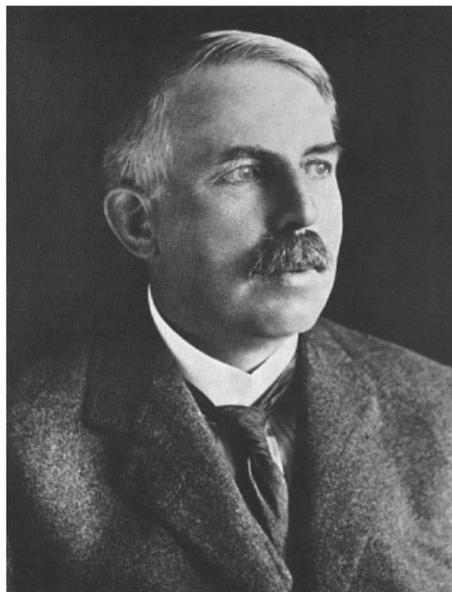


Methods based on Sylvester–Uhlmann 87 CGO solutions

- ▶ countable family  $(u_j^i, A_{u_j^i})_{j=1}^{\infty}$  determines  $V$

# What about in physics?

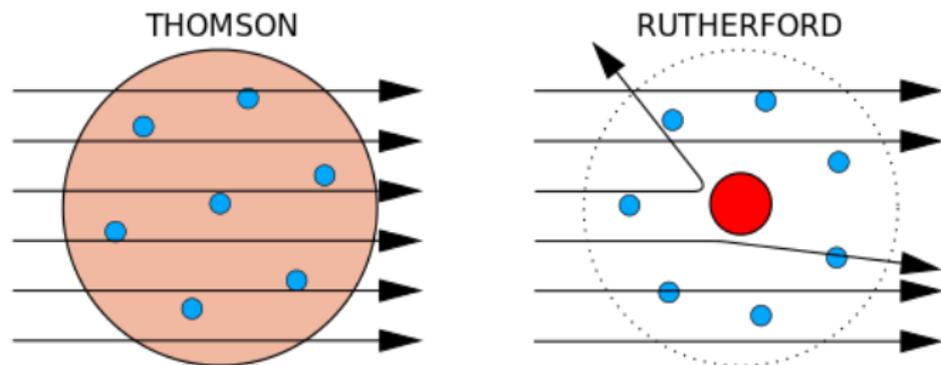
Lord Rutherford's gold-foil experiment



Single incident wave

# Scattering theory

## Rutherford experiment's conclusions



measurement + *a-priori information* = conclusion

# Corner scattering

Theorem (B.-Päivärinta–Sylvester 14)

*The potential  $V = \chi_{[0, \infty[^n} \varphi$ ,  $\varphi(0) \neq 0$  always scatters.*

# Corner scattering

Theorem (B.–Päivärinta–Sylvester 14)

*The potential  $V = \chi_{[0,\infty[^n}\varphi$ ,  $\varphi(0) \neq 0$  always scatters.*

For *any* incident wave  $u^i \neq 0$  we have  $A_{V,u^i} \neq A_{0,u^i}$ .

## Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int Vu' u_0 dx = 0$$

if  $(\Delta + k^2(1 + V))u_0 = 0$  near  $\text{supp } V$ .

## Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if  $(\Delta + k^2(1 + V))u_0 = 0$  near  $\text{supp } V$ .

In simple case

$$u^i(x) = u^i(0) + u_r^i(x)$$

$$u_0(x) = e^{\rho \cdot x} (1 + \psi(x))$$

$$V(x) = \chi_{[0, \infty[^n}(x) (\varphi(0) + \varphi_r(x))$$

## Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if  $(\Delta + k^2(1 + V))u_0 = 0$  near  $\text{supp } V$ .

In simple case

$$u^i(x) = u^i(0) + u_r^i(x)$$

$$u_0(x) = e^{\rho \cdot x} (1 + \psi(x))$$

$$V(x) = \chi_{[0, \infty[^n}(x) (\varphi(0) + \varphi_r(x))$$

Hölder estimates give

$$C \left| \varphi(0) u^i(0) \right| |\rho|^{-n} \leq \left| \varphi(0) u^i(0) \int_{[0, \infty[^n} e^{\rho \cdot x} dx \right| \leq C |\rho|^{-n-\delta}$$

if  $\|\psi\|_p \leq C |\rho|^{-n/p-\varepsilon}$ .

## Some newer corner scattering results

- ▶ Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- ▶ Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- ▶ Elschner–Hu: 3D any domain having two faces meet at an angle
- ▶ Liu–Xiao: electromagnetic waves

## Some newer corner scattering results

- ▶ Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- ▶ Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- ▶ Elschner–Hu: 3D any domain having two faces meet at an angle
- ▶ Liu–Xiao: electromagnetic waves

Injectivity of support probing:

### Theorem (HSV+EH)

Let  $P, P'$  be convex polyhedra and  $V = \chi_P \varphi$ ,  $V = \chi_{P'} \varphi'$  for admissible functions  $\varphi, \varphi'$ . Then

$$P \neq P' \implies A_{u^i} \neq A'_{u^i} \quad \forall u^i \neq 0$$

Any *single* incident wave determines  $P$  in the class of polyhedral penetrable scatterers.

# Stability of polygonal scatterer probing

Non-vanishing total wave

Theorem (B., Liu, preprint)

Let  $u^i$  be an incident wave and let  $V = \chi_P \varphi$ ,  $V' = \chi_{P'} \varphi'$  be admissible with  $|u|, |u'| \neq 0$  in  $B_R$ . If

$$\|A_{u^i} - A'_{u^i}\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C(\ln \ln \|A_{u^i} - A'_{u^i}\|_2^{-1})^{-\eta}$$

for some  $\eta > 0$ .

# Stability of polygonal scatterer probing

Non-vanishing total wave

Theorem (B., Liu, preprint)

Let  $u^i$  be an incident wave and let  $V = \chi_P \varphi$ ,  $V' = \chi_{P'} \varphi'$  be admissible with  $|u|, |u'| \neq 0$  in  $B_R$ . If

$$\|A_{u^i} - A'_{u^i}\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C(\ln \ln \|A_{u^i} - A'_{u^i}\|_2^{-1})^{-\eta}$$

for some  $\eta > 0$ .

Related work: Probing [impenetrable](#) scatterers with few waves:  
J. Li, H. Liu, M. Petrini, L. Rondi, J. Xiao, Y. Wang ...

# Lower bound for far-field pattern

Arbitrary Herglotz wave

Theorem (B., Liu, preprint)

Let  $u^i$  be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta), \quad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let  $V = \chi_P \varphi$  be admissible. Then

$$\|A_{u^i}\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|, V} > 0$$

where the Taylor expansion of  $u^i$  at the corner  $x_c$  begins with  $P_N$ , and  $\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)| d\sigma(\theta)$ .

## Mistake?



F. Cakoni: “Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns.”

## From apparent contradiction to inspiration

Theorem (B., Liu, + B., Li, Liu, Wang, preprints)

Let the potential  $V = \chi_P \varphi$  be admissible and  $P \subset \Omega$ . Let  $v$  be a transmission eigenfunction:

$$\begin{aligned}(\Delta + k^2)v &= 0, & \Omega \\(\Delta + k^2(1 + V))w &= 0, & \Omega \\w - v &\in H_0^2(\Omega), & \|v\|_{L^2(\Omega)} = 1.\end{aligned}$$

If  $v$  can be approximated by a sequence of Herglotz waves with uniformly  $L^2$ -bounded kernels  $g$ , then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x_c, r))} \int_{B(x_c, r)} |v(x)| \, dx = 0$$

at every corner point  $x_c$  of  $\text{supp } V$ .

## Piecewise constant recovery

Injectivity of piecewise constant potential probing:

Theorem (B., Liu, preprint)

Let  $\Sigma_j, j = 1, 2, \dots$  be bounded convex polyhedra in an admissible geometric arrangement (think pixels/voxels) and  $V = \sum_j V_j \chi_{\Sigma_j}$ ,  $V' = \sum_j V'_j \chi_{\Sigma_j}$  for constants  $V_j, V'_j \in \mathbb{C}$ . Then

$$V \neq V' \implies A_{u^i} \neq A'_{u^i} \quad \forall u^i(x) = e^{ik\theta \cdot x}$$

if  $k > 0$  small or  $|u| + |u'| \neq 0$  at each vertex.

A single incident plane wave determines  $V$  in the class of discretized penetrable scatterers.

## Proof sketch

Integration by parts

$$k^2 \int_{\Omega} (V - V') u' u_0 dx = \int_{\partial\Omega} ((u - u') \partial_{\nu} u_0 - u_0 \partial_{\nu} (u - u')) dx$$

if  $(\Delta + k^2(1 + V))u_0 = 0$  in  $\Omega$ .

## Proof sketch

Integration by parts

$$k^2 \int_{\Omega} (V - V') u' u_0 dx = \int_{\partial\Omega} ((u - u') \partial_{\nu} u_0 - u_0 \partial_{\nu} (u - u')) dx$$

if  $(\Delta + k^2(1 + V))u_0 = 0$  in  $\Omega$ .

Simple case:  $\Omega = B(0, \varepsilon) \cap \Sigma_j$  with  $\Sigma_j = ]0, 1[{}^n$

$$u'(x) = u'(0) + u'_r(x) \quad u' \in H^2 \hookrightarrow C^{1/2}$$

$$u_0(x) = e^{\rho \cdot x} (1 + \psi(x)) \quad \text{CGO}$$

$$(V - V')(x) = V_j - V'_j \quad \text{piecewise constant}$$

## Proof sketch

Integration by parts

$$k^2 \int_{\Omega} (V - V') u' u_0 dx = \int_{\partial\Omega} ((u - u') \partial_{\nu} u_0 - u_0 \partial_{\nu} (u - u')) dx$$

if  $(\Delta + k^2(1 + V))u_0 = 0$  in  $\Omega$ .

Simple case:  $\Omega = B(0, \varepsilon) \cap \Sigma_j$  with  $\Sigma_j = ]0, 1[^n$

$$u'(x) = u'(0) + u'_r(x) \quad u' \in H^2 \hookrightarrow C^{1/2}$$

$$u_0(x) = e^{\rho \cdot x} (1 + \psi(x)) \quad \text{CGO}$$

$$(V - V')(x) = V_j - V'_j \quad \text{piecewise constant}$$

Hölder estimates give

$$C |(V_j - V'_j) u'(0)| |\rho|^{-n} \leq |(V_j - V'_j) u'(0)| \int_{[0, \infty[^n} e^{\rho \cdot x} dx \leq C |\rho|^{-n-\delta}$$

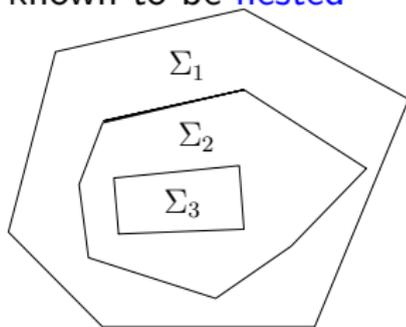
if  $\|\psi\|_{\rho} \leq C |\rho|^{-n/p-\varepsilon}$ .

## Generalizations and limitations

- ▶ unique determination of corner location *and* value

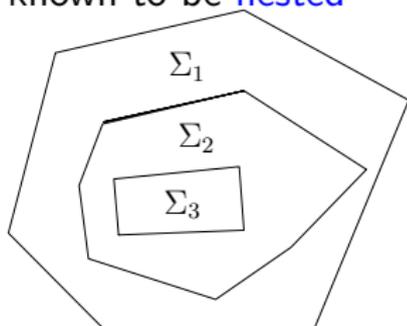
## Generalizations and limitations

- ▶ unique determination of corner location *and* value
- ▶ if  $\Sigma_j$  not known in advance: both  $(\Sigma_j)_{j=1}^{\infty}$  and  $V = \sum_j V_j \chi_{\Sigma_j}$  uniquely determined by a single measurement if geometry known to be **nested**

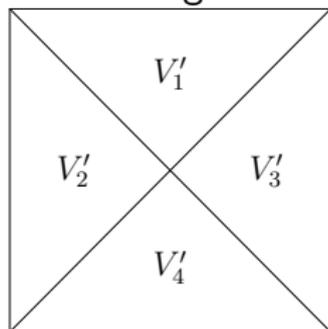
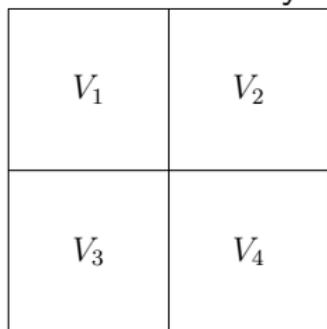


## Generalizations and limitations

- ▶ unique determination of corner location *and* value
- ▶ if  $\Sigma_j$  not known in advance: both  $(\Sigma_j)_{j=1}^{\infty}$  and  $V = \sum_j V_j \chi_{\Sigma_j}$  uniquely determined by a single measurement if geometry known to be **nested**



- ▶ method cannot yet be shown to distinguish between



Thank you for your attention!