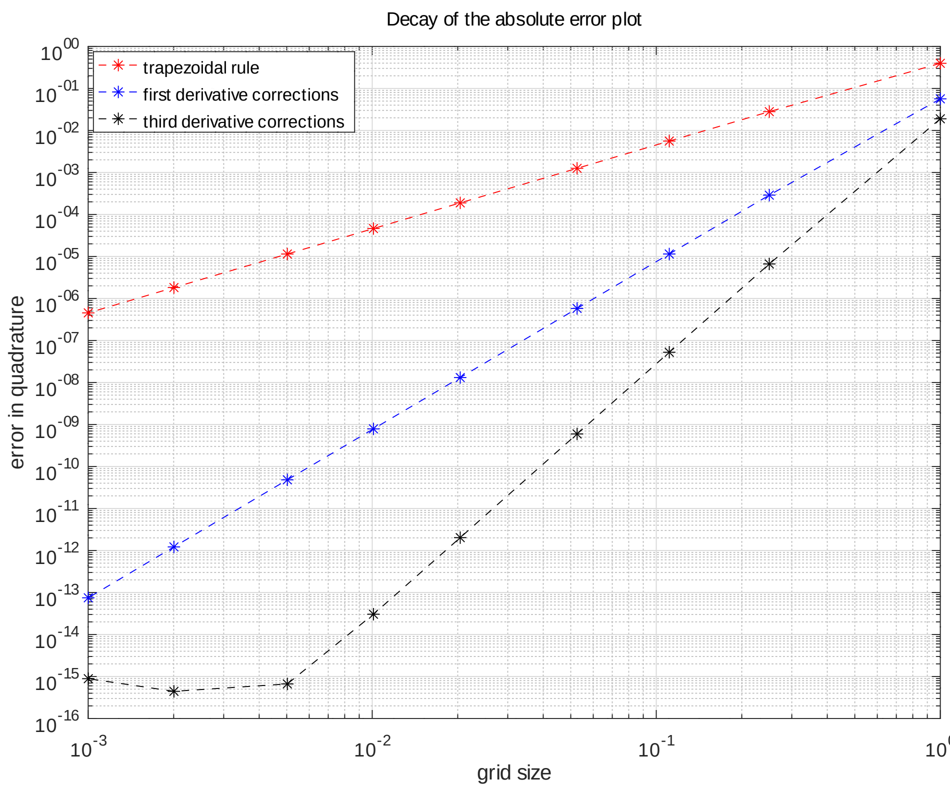
1. Compute $\int_0^1 e^{x^2}\, dx$ using the trapezoidal rule and trapezoidal rule with end corrections using the first and third derivatives. Perform this by subdividing $[0,1]$ into $N \in \{2, 5, 10, 20, 50, 100, 200, 500, 1000\}$ panels and plot the decay of the absolute error using the **three methods**. The value of the integral accurate upto 16 digits is 1.4626517459071816.

   Program:

```
#!/usr/bin/env octave
% File: quadrature.m
% Script to compute the integral using the trapezoidal rule

% end points of interval
a = 0; b = 1;

% function to be integrated
f = @(x) exp(x.^2);

% first derivative of the function
g = @(x) 2*x*exp(x.^2);

% third derivative of the function
s = @(x) 12*x*exp(x^2) + 8*x^3*exp(x^2);

% exact value of the integral
exact = 1.4626517459071816;

% number of grid points
N = [2, 5, 10, 20, 50, 100, 200, 500, 1000];

% number of different set of grids
Ngrids = length(N);

h = zeros(Ngrids,1);          % different grid spacings

trap = zeros(Ngrids,1);       % trapezoidal rule
tenf = zeros(Ngrids,1);       % trapezoidal rule with end corrections using first derivative
tent = zeros(Ngrids,1);       % trapezoidal rule with end corrections using third derivative

for k = 1:Ngrids
    h(k) = (b - a)/(N(k) - 1);                 % grid spacing
    x = linspace(a, b, N(k));                  % grid points
    y = 0.5*(x(1:N(k)-1) + x(2:N(k)));         % mid-points of each panel

    trap(k) = h(k)*(sum(f(x)) - (f(a) + f(b))/2);         % trapezoidal rule
    tenf(k) = trap(k) - (h(k)^2/12) * (g(b) - g(a));      % first derivative corrections
    tent(k) = tenf(k) + (h(k)^4/720) * (s(b) - s(a));     % third derivative corrections
end

% error calculations
trap_err = abs(double(trap - exact));    % error in trapezoidal rule
tenf_err = abs(double(tenf - exact));    % error in first end corrections
tent_err = abs(double(tent - exact));    % error in third end corrections

figure;  loglog(h, trap_err, 'r*--'); hold on; loglog(h, tenf_err, 'b*--');
hold on; loglog(h, tent_err, 'k*--'); xlabel('grid size'); ylabel('error in quadrature');
legend('trapezoidal rule', 'first derivative corrections', 'third derivative corrections');
set(gca, 'title', text('string', 'Decay of the absolute error plot'));
set(legend, 'location', 'northwest'); grid on; print('quadrature.pdf', '-dpdf')
```



2. Use the Euler-Macluarin to obtain

$$\log(n!) = \log\left(C\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \mathcal{O}(1/n)$$

where C is some constant.

Solution:

The Euler-Maclaurin summation formula:

$$\sum_{n=0}^{\infty} f(x+n) = \int_{x}^{\infty} f(x)\ dx + \frac{1}{2}f(x) - \sum_{n=2}^{\infty} \frac{B_n}{n!}\frac{d^{n-1}f(x)}{dx^{n-1}} \tag{1}$$

It can be rewritten for the case of a finite sum as following:

$$\begin{aligned}
\sum_{k=n}^{N} f(k) &= \sum_{k=n}^{\infty} f(k) - \sum_{k=N+1}^{\infty} f(k) &(2)\\
&= \sum_{k=n}^{\infty} f(k) - \sum_{k=N}^{\infty} f(k) + f(N) &(3)\\
&= \sum_{k=0}^{\infty} f(k+n) - \sum_{k=0}^{\infty} f(k+N) + f(N) &(4)\\
&= \int_{n}^{\infty} f(x)\ dx - \int_{N}^{\infty} f(x)\ dx + \frac{1}{2}f(n) - \frac{1}{2}f(N) + f(N) + \sum_{n=2}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}f(n)}{dx^{n-1}} + \sum_{n=2}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}f(N)}{dx^{n-1}} &(5)\\
&= \int_{n}^{N} f(x)\ dx + \frac{1}{2}\left[f(n) + f(N)\right] + \sum_{n=2}^{\infty}\frac{B_n}{n!}\left[\frac{d^{n-1}f}{dx^{n-1}}\bigg|_{x=N} - \frac{d^{n-1}f}{dx^{n-1}}\bigg|_{x=n}\right] &(6)
\end{aligned}$$

let's consider the Stirling's approximation for $\Gamma(n)$ for positive integer $n \gg 1$

$$\Gamma(n+1) = n! \tag{7}$$

$$\ln(\Gamma(n+1)) = \ln n! = \ln(1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n) = \sum_{k=1}^{n} \ln k \tag{8}$$

The first two terms in Eq.(6) give us the following approximation:

$$\ln(\Gamma(n+1)) = \ln(n!) = \int_{1}^{n} \ln(x)\ dx + \frac{1}{2}(\ln(1) + \ln(n)) = x\ln(x)\bigg|_{1}^{n} - \int_{1}^{n} \frac{x}{x}\ dx + \frac{1}{2}\ln(n) = n\ln(n) - n + 1 + \frac{1}{2}\ln(n)$$

Let's notice first that the term with the derivatives of $\ln(x)$ at $x = n$ in Eq.(6) are proportional to negative powers of n and thus $\to 0$ as $n \to \infty$. On the other hand, the sum of the term with the derivatives of $\ln(x)$ at $x = 1$ is a constant independent of n. Thus,

$$\begin{aligned}
\ln\Gamma(n+1) &= \ln n!\\
&= \ln\left(\frac{n}{e}\right)^n + \ln\sqrt{n} + \ln(C)\\
&= \ln\left(C\sqrt{n}\left(\frac{n}{e}\right)^n\right)
\end{aligned}$$

3. We will now determine C in the above question as follows.

- Use integration by parts to obtain an expression for $I_k = \int_0^{\pi/2} sin^k(x)\ dx$ (It might be easier to look at the even and odd cases separately)
- Prove that I_k is a monotone decreasing sequence.
- Show that

$$\lim_{m\to\infty}\frac{I_{2m-1}}{I_{2m+1}} = 1$$

- Conclude that
$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1$$
- Hence, infer that the central binomial coefficient is asymptotically given by
$$\binom{2m}{m} \sim \frac{4^m}{\sqrt{m\pi}}$$
where $f(m) \sim g(m) \implies \lim_{m\to\infty} \dfrac{f(m)}{g(m)} = 1$
- Conclude that C in the above question is $\sqrt{2\pi}$
- Hence, obtain the Stirling formula:
$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$
- Obtain the relative error in $n!$ using the Stirling formula for $n \in \{20, 50\}$
- Obtain a better estimate for $n!$, which is accurate upto $\mathcal{O}(1/n^3)$