1. We build a computer, where the real numbers are represented using 5 digits as explained below:

S	A	B	C	E

where

- S is the sign bit; 0 is positive and 1 is negative
- A, B, C: First three significant digits in decimal expansion with decimal point occuring between A and B
- E is the exponent in base 10 with a bias of 5
- All digits after the third significant digit are chopped off
- $+0$ is represented by setting $S = 0$ and $A = 0$; B, C, E can be anything
- -0 is represented by setting $S = 1$ and $A = 0$; B, C, E can be anything
- $+\infty$ is represented by setting $S = 0$ and $A = B = C = E = 9$
- $-\infty$ is represented by setting $S = 1$ and $A = B = C = E = 9$
- Not A Number is represented by setting S to be other than 0 and 1.

For example, the number $\pi = 3.14159...$ is represented as follows. Chopping off after the third significant digit, we have $\pi = +3.14 \times 10^0$. Hence, the representation of π on our machine is:

0	3	1	4	5

The number $-0.001259...$ is represented as follows. Chopping off after the third significant digit, we have -1.25×10^{-3}. Hence, the representation of on our machine is:

1	1	2	5	2

Now answer the following questions:

(a) How many non-zero Floating Point Numbers (from now on abbreviated as FPN) can be represented by our machine?

Solution:
The representation of $-\infty$ on our machine is:

1	9	9	9	9

$True\ exponent = E - bias = 9 - 5 = 4$
With decimal point occuring between A and B, the decimal point can float upto 4 places to the right, which implies that $-\infty = -99900$

The next smallest negative floating point number represented on this machine is -99899. Chopping off after the third significant, we have -9.98×10^4

$E = True\ exponent + bias = 4 + 5 = 9$
Hence, the representation of on our machine is:

1	9	9	8	9

The representation of $+\infty$ on our machine is:

0	9	9	9	9

$True\ exponent = E - bias = 9 - 5 = 4$
With decimal point occuring between A and B, the decimal point can float upto 4 places to the right, which implies that $+\infty = 99900$

The next largest positive floating point number represented on this machine is +99899. Chopping off after the third significant, we have $+9.98 \times 10^4$

$E = True\ exponent + bias = 4 + 5 = 9$
Hence, the representation of on our machine is:

0	9	9	8	9

(b) How many FPN are in the following intervals?

- $(9, 10)$

  Solution:
  The smallest number exceeding 9 that can be represented exactly on this machine is 9.01

0	9	0	1	5

  The largest number not exceeding 10 that can be represented exactly on this machine is 9.99

0	9	9	9	5

  Hence, there are 99 FPN in the interval $(9, 10)$

- $(10, 11)$

  Solution:
  The smallest number exceeding 10 that can be represented exactly on this machine is 10.1

0	1	0	1	6

  The largest number not exceeding 11 that can be represented exactly on this machine is 10.9

0	1	0	9	6

  Hence, there are 9 FPN in the interval $(10, 11)$

- $(0, 1)$

  Solution:
  The smallest number exceeding 0 that can be represented exactly on this machine is 0.00001

0	1	0	0	0

  The largest number not exceeding 1 that can be represented exactly on this machine is 0.999

0	9	9	9	4

(c) Identify the smallest positive and largest positive FPN on the machine

Solution:
Smallest positive floating point number:

$$S = 0 \qquad E = 0 \qquad ABC = 100$$

$$True\ exponent = E - bias = 0 - 5 = -5$$

$$10^{-5} = 0.00001$$

0	1	0	0	0

Largest positive floating point number:

$$S = 0 \qquad E = 9 \qquad M = 998$$

$$True\ exponent = E - bias = 9 - 5 = 4$$

$$9.98 \times 10^4 = 99800.0$$

0	9	9	8	9

(d) Identify the machine precision

Solution:
Machine precision is defined as the difference between the smallest number exceeding 1 that can be represented on the machine and 1. The smallest number exceeding 1 that can be represented on this machine is $1.01 = 1 + 10^{-2}$
Hence, the machine precision is $\epsilon_m = 10^{-2} = 0.01$

(e) What is the smallest positive integer not representable exactly on this machine?

(f) Consider solving the following recurrence on our machine:

$$a_{n+1} = 5a_n - 4a_{n-1}$$

with $a_1 = a_2 = 2.932$. Compute a_n for $n \epsilon\ 3, 4, 5, 6, 7$ on our machine (work out what the machine would do by hand). Note a_1, a_2 would be chopped to three significant digits to begin with. Next note that at each step in the recurrence $5a_n$ and $4a_{n-1}$ would be chopped down to the first three significant digits before the subtraction is performed.

Solution:

$$a_{n+1} = 5a_n - 4a_{n-1}$$

0	2	9	3	5

Compute a_3: $a_1 = a_2 = 2.93$

$$\begin{aligned} a_3 &= 5a_2 - 4a_1 \\ &= 5(2.93) - 4(2.93) \\ &= 14.65 - 11.72 \\ &= 1.46 \times 10^1 - 1.17 \times 10^1 \\ &= 0.29 \times 10^1 = 2.90 \end{aligned}$$

0	2	9	0	5

Compute a_4: $a_2 = 2.93,\ a_3 = 2.90$

$$\begin{aligned} a_4 &= 5a_3 - 4a_2 \\ &= 5(2.90) - 4(2.93) \\ &= 14.50 - 11.72 \\ &= 1.45 \times 10^1 - 1.17 \times 10^1 \\ &= 0.28 \times 10^1 = 2.80 \end{aligned}$$

0	2	8	0	5

Compute a_5: $a_3 = 2.90,\ a_4 = 2.80$

$$\begin{aligned} a_5 &= 5a_4 - 4a_3 \\ &= 5(2.80) - 4(2.90) \\ &= 14.0 - 11.6 \\ &= 1.40 \times 10^1 - 1.16 \times 10^1 \\ &= 0.24 \times 10^1 = 2.40 \end{aligned}$$

0	2	4	0	5

Compute a_6: $a_4 = 2.80, a_5 = 2.40$

$$\begin{aligned} a_6 &= 5a_5 - 4a_4 \\ &= 5(2.40) - 4(2.80) \\ &= 12.0 - 11.2 \\ &= 1.20 \times 10^1 - 1.12 \times 10^1 \\ &= 0.08 \times 10^1 = 8.0 \times 10^{-1} \end{aligned}$$

0	8	0	0	4

Compute a_7: $a_5 = 2.40, a_6 = 0.80$

$$\begin{aligned} a_7 &= 5a_6 - 4a_5 \\ &= 5(0.8) - 4(2.40) \\ &= 4.0 - 9.6 = -5.6 \end{aligned}$$

1	5	6	0	5

2. Consider the following integral:

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

(a) Obtain a recurrence for I_n in terms of I_{n-1}. (HINT: Integration by parts)

Solution:

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

$$I_{n-1} = \int_0^1 x^{2(n-1)} \sin(\pi x)\, dx$$

Integration by parts:

$$\int u\, dv = uv - \int v\, du$$

$$u = x^{2n} \qquad dv = \sin(\pi x)\, dx$$

$$du = 2nx^{2n-1}dx \qquad v = -\frac{1}{\pi}\cos(\pi x)$$

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx = uv\Big|_0^1 - \int_0^1 v\, du$$

$$= -\frac{1}{\pi}x^{2n}cos(\pi x)\Big|_0^1 + \frac{2n}{\pi}\int_0^1 x^{2n-1}cos(\pi x)\, dx$$

$$= -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi}\int_0^1 x^{2n-1}cos(\pi x)\, dx$$

$$u = x^{2n-1} \qquad dv = \cos(\pi x)\, dx$$

$$du = (2n-1)x^{2n-2}dx \qquad v = \frac{1}{\pi}\sin(\pi x)$$

$$= -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi}\left[\frac{1}{\pi}x^{2n-1}sin(\pi x)\Big|_0^1 - \frac{2n-1}{\pi}\int_0^1 x^{2n-2}sin(\pi x)\, dx\right]$$

$$= -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi}\left[\frac{1}{\pi}1^{2n-1}sin(\pi) - \frac{2n-1}{\pi}\int_0^1 x^{2n-2}sin(\pi x)\, dx\right]$$

$$I_n = -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi^2}1^{2n-1}\sin(\pi) - \frac{2n(2n-1)}{\pi^2}I_{n-1}$$

(b) Evaluate I_0 by hand

Solution:

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

$$I_0 = \int_0^1 x^0 \sin(\pi x)\, dx$$

$$I_0 = \int_0^1 \sin(\pi x)\, dx$$

$$= \left[\frac{1}{\pi}(-\cos \pi x)\right]_0^1$$

$$= \frac{1}{\pi}(-\cos \pi) - \frac{1}{\pi}(-\cos \pi)$$

$$= \frac{1}{\pi}(1+1) = \frac{2}{\pi} \approx 0.63662$$

(c) Use the recurrence to obtain I_n for $n \in 1, 2, ..., 15$ in Octave

```
#!/usr/bin/env octave
% File: recurrence.m
% Script to obtain I(n) by recurrence relation

N = 15;              % number of terms in the recurrence
I = zeros(N,1);      % initialize vector of N elements
I(1) = 0.63662;      % initial condition I(1) = 0.63662

for n = 1:N
    I(n+1) = (-1/pi) * 1^(2*n) * cos(pi) \
             + (2*n)/(pi^2) * 1^(2*n - 1) * sin(pi) \
             - ((2*n)*(2*n - 1)/(pi^2)) * I(n);
endfor

disp(I)
```

Output:

```
 6.3662e-01
 1.8930e-01
 8.8144e-02
 5.0384e-02
 3.2433e-02
 2.2552e-02
 1.6692e-02
 1.0503e-02
 6.2906e-02
-1.6320e+00
 6.3155e+01
-2.9560e+03
 1.6533e+05
-1.0888e+07
 8.3402e+08
-7.3519e+10
```

(d) Use wolframalpha to obtain I_n by directly performing the integeral for $n \in 1, 2, ..., 15$

Solution:

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

$n = 1$ $$I_1 = \int_0^1 x^{2(1)} \sin(\pi x) dx \approx 0.18930$$

$n = 2$ $$I_2 = \int_0^1 x^{2(2)} \sin(\pi x) dx \approx 0.088144$$

$n = 3$ $$I_3 = \int_0^1 x^{2(3)} \sin(\pi x) dx \approx 0.050384$$

$n = 4$ $$I_4 = \int_0^1 x^{2(4)} \sin(\pi x) dx \approx 0.032433$$

$n = 5$ $$I_5 = \int_0^1 x^{2(5)} \sin(\pi x) dx \approx 0.022561$$

$$n = 6 \qquad I_6 = \int_0^1 x^{2(6)} \sin(\pi x) dx \approx 0.016574$$

$$n = 7 \qquad I_7 = \int_0^1 x^{2(7)} \sin(\pi x) dx \approx 0.012679$$

$$n = 8 \qquad I_8 = \int_0^1 x^{2(8)} \sin(\pi x) dx \approx 0.010006$$

$$n = 9 \qquad I_9 = \int_0^1 x^{2(9)} \sin(\pi x) dx \approx 0.0080938$$

$$n = 10 \qquad I_{10} = \int_0^1 x^{2(10)} \sin(\pi x) dx \approx 0.0066802$$

$$n = 11 \qquad I_{11} = \int_0^1 x^{2(11)} \sin(\pi x) dx \approx 0.0056060$$

$$n = 12 \qquad I_{12} = \int_0^1 x^{2(12)} \sin(\pi x) dx \approx 0.0047708$$

$$n = 13 \qquad I_{13} = \int_0^1 x^{2(13)} \sin(\pi x) dx \approx 0.0041089$$

$$n = 14 \qquad I_{14} = \int_0^1 x^{2(14)} \sin(\pi x) dx \approx 0.0035754$$

$$n = 15 \qquad I_{15} = \int_0^1 x^{2(15)} \sin(\pi x) dx \approx 0.0031393$$

(e) Explain your observation

The values obtained from the octave script recurrence relation and the integral calculated are same from $n = 0$ to $n = 4$ and differs for other values of n.