Numerical Methods and Scientific Computing

End Semester Exam

1. (3 points) An integral of the form $\int_0^1 f(x)dx$ was computed by two students using trapezoidal rule with end point corrections involving only the first derivative. The first student reported the value as 0.8 using a grid spacing of h, while the second student reported the value as 0.75 using a grid spacing of $h/2$. A smart, lazy student from NMSC class enters their discussion and gives a better answer of the integral with a higher order of accuracy by processing the information above. How did he do it? What was his answer? What is the order of accuracy of his answer?

2. (10 points) Gaussian quadrature:

   - (4 points) Find the first three monic polynomials (i.e., till quadratic) on $[0, 1]$ orthogonal with respect to the inner product
   $$\langle f, g\rangle = \int_0^1 \frac{x}{\sqrt{1-x^2}} f(x)g(x)dx$$
   - (2 points) Use the above to find a quadrature formula of the form
   $$\int_0^1 \frac{x}{\sqrt{1-x^2}} f(x)dx = \sum_{i=0}^{n} a_i f(x_i)$$
   that is exact for all $f(x)$ of degree 3.
   - (4 points) Use the above to evaluate $\int_0^1 \frac{x\sin(x)}{\sqrt{1-x^2}}dx$

3. (3 points) Comment on using the Newton method to compute the root of the function $f(x) = x^{1/3}$, i.e., if your initial guess is $x_0 = 1$, what would be the value of x_n? Does the method converge to the root we want?

4. (6 points) It is given that a sequence of Newton iterates converge to a root r of the function $f(x)$. Further, it is given that the root r is a root of multiplicity 2, i.e., $f(x) = (x - r)^2 g(x)$, where $g(r) \neq 0$. It is also given that the function f, its derivatives till the second order are continuous in the neighbourhood of the root r. If e_n is the error of the n^{th} iterate, i.e., $e_n = x_n - r$, then obtain
$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n}$$

5. (4 points) **Bonus question**: What happens to the above if the root r has a multiplicity m?

6. (16 points) Compute $\int_{-1}^{1} e^{-x^2} dx$ using the

(a) Trapezoidal rule

(b) Trapezoidal rule with end corrections using the first derivative

(c) Trapezoidal rule with end corrections using the first derivative and third derivatives

(d) Gauss-Legendre quadrature

- Perform this by subdividing $[-1, 1]$ into $N \in \{2, 5, 10, 20, 50, 100\}$ panels
- Plot the decay of the absolute error using the above methods.
- You may obtain the exact value of the integral upto 20 digits using wolframalpha.
- Make sure the figure has a legend and the axes are clearly marked.
- Ensure that the font size for title, axes, legend are readable.
- Submit the plots obtained, entire code and the write-up.

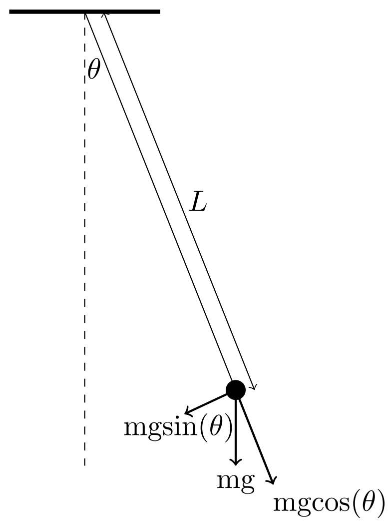
7. (8 points) Evaluate $I = \int_0^1 \frac{e^{-x}}{\sqrt{x}} dx$ by subdividing the domain into $N \in \{5, 10, 20, 50, 100, 200, 500, 1000\}$ panels.

(a) Using a rectangular rule

(b) Make a change of variables $x = t^2$ and use rectangular rule on new variable.

- Plot the decay of the absolute error using the above two methods.
- You may obtain the exact value of the integral upto 20 digits using wolframalpha.
- Compare the two methods above in terms of accuracy and cost.
- Explain the difference in solution, if any.
- Make sure the figure has a legend and the axes are clearly marked.
- Ensure that the font size for title, axes, legend are readable.
- Submit the plots obtained, entire code and the write-up.

8. (20 points) Consider the motion of a simple pendulum



The restoring force is $mg \sin\theta$ and hence the governing equation is

$$mL\frac{d^2\theta}{dt^2} + mg\sin(\theta) = 0$$

Let the length of the string be g. Hence, the governing equation simplifies to

$$\frac{d^2\theta}{dt^2} + \sin(\theta) = 0$$

At the initial time, the pendulum is pulled to an angle of $\theta = 30° = \frac{\pi}{6}$ before being let loose without any velocity imparted. Write a code to solve for the motion of the pendulum till $t = 100$ seconds using

(a) Forward Euler

(b) Backward Euler

(c) Trapezoidal Rule

- Recall that you need to need to reformulate the second order differential equation as a system of first order differential equation.
- Vary your time step Δt in $\{0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20\}$.
- For each Δt plot the solution obtained by the three methods on a separate figure till the final time of 100.
- Discuss the stability of the schemes. From your plots, at what Δt do these schemes become unstable (if at all they become unstable)?

- Analyse the stability of the three numerical methods to solve the differential equation by approximating $\sin(\theta)$ to be θ.
- Make sure each figure has a legend and the axes are clearly marked.
- Ensure that the font size for title, axes, legend are readable.
- Submit the plots obtained, entire code and the write-up.