1. Consider the Vandermonde matrix V, i.e.,

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & . & . & . & x_0^n \\ 1 & x_1 & x_1^2 & . & . & . & x_1^n \\ 1 & x_2 & x_2^2 & . & . & . & x_2^n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_n & x_n^2 & . & . & . & x_n^n \end{bmatrix}$$

(a) Show that $det(V)$ is a polynomial in the variables $x_0, x_1, ..., x_n$ with degree $\dfrac{n(n+1)}{2}$

Solution:
The Vandermonde determinant $det(V) \neq 0$ if and only if the variables $x_0, x_1, ..., x_n$ are distinct.
Let $n = 2$, then,

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

$$det(V) = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}$$

$$\begin{aligned}
&= 1(x_1x_2^2 - x_1^2x_2) - x_0(x_2^2 - x_1^2) + x_0^2(x_2 - x_1) \\
&= x_1x_2^2 - x_1^2x_2 - x_0x_2^2 + x_0x_1^2 + x_0^2x_2 - x_1x_0^2 \\
&= x_1x_2^2 - x_1^2x_2 - x_0x_2^2 + x_0x_1x_2 - x_1x_2x_0 + x_1^2x_0 + x_0^2x_2 - x_0^2x_1 \\
&= (x_1x_2 - x_1^2 - x_0x_2 + x_0x_1)(x_2 - x_0) \\
&= (x_1 - x_0)(x_2 - x_1)(x_2 - x_0) \\
&= \prod_{0 \leq j < i \leq 2} (x_i - x_j)
\end{aligned}$$

(b) Show that if $x_i = x_j$ for $i \neq j$, then $det(V) = 0$

Solution:
The Vandermonde determinant $det(V) \neq 0$ if and only if the variables $x_0, x_1, ..., x_n$ are distinct.
Let $n = 1$, then,

$$det(V) = \begin{vmatrix} 1 & x_0 \\ 1 & x_1 \end{vmatrix}$$

$$\begin{aligned}
&= (x_1 - x_0) \\
&= 0 \qquad if \;\; x_1 = x_0
\end{aligned}$$

In general, for degree n,

$$\begin{aligned}
det(V) &= \prod_{0 \leq j < i \leq n} (x_i - x_j) \\
&= 0 \qquad if \;\; x_i = x_j
\end{aligned}$$

(c) Conclude that $(x_i - x_j)$ is a factor of $det(V)$

Solution: Since $det(V)$ is a polynomial, this implies that $(x_i - x_j)$ is a factor of $det(V)$.

(d) Conclude that $det(V) = C\left(\prod_{0\le j<i\le n}(x_i - x_j)\right)$, where C is a constant

Solution:

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & . & . & . & x_0^n \\ 1 & x_1 & x_1^2 & . & . & . & x_1^n \\ 1 & x_2 & x_2^2 & . & . & . & x_2^n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_n & x_n^2 & . & . & . & x_n^n \end{bmatrix}$$

$$det(V) = det\begin{bmatrix} 1 & x_0 & x_0^2 & . & . & . & x_0^n \\ 1 & x_1 & x_1^2 & . & . & . & x_1^n \\ 1 & x_2 & x_2^2 & . & . & . & x_2^n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_n & x_n^2 & . & . & . & x_n^n \end{bmatrix}$$

then, by row operations we get

$$= det\begin{bmatrix} 1 & x_0 & x_0^1 & . & . & . & x_0^{n-1} & x_0^n \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & . & . & . & x_1^{n-1} - x_0^{n-1} & x_1^n - x_0^n \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & x_n - x_0 & x_n^2 - x_0^2 & . & . & . & x_n^{n-1} - x_0^{n-1} & x_n^n - x_0^n \end{bmatrix}_{n+1\times n+1}$$

$$= det\begin{bmatrix} x_1 - x_0 & x_1^2 - x_0^2 & . & . & . & x_1^{n-1} - x_0^{n-1} & x_1^n - x_0^n \\ x_2 - x_1 & x_2^2 - x_1^2 & . & . & . & x_2^{n-1} - x_1^{n-1} & x_2^n - x_1^n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ x_n - x_0 & x_n^2 - x_0^2 & . & . & . & x_n^{n-1} - x_0^{n-1} & x_n^n - x_0^n \end{bmatrix}_{n\times n}$$

$$= det\begin{bmatrix} x_1 - x_0 & 0 & . & . & . & 0 \\ 0 & x_2 - x_0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & x_n - x_0 \end{bmatrix}\begin{bmatrix} 1 & x_1 + x_0 & . & . & . & \sum_{i=0}^{n-1} x_1^{n-1-i}x_0^i \\ 1 & x_2 + x_0 & . & . & . & \sum_{i=0}^{n-1} x_2^{n-1-i}x_0^i \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & x_n + x_0 & . & . & . & \sum_{i=0}^{n-1} x_n^{n-1-i}x_0^i \end{bmatrix}$$

and, by Cauchy's Theorem, we get that

$$= \prod_{j=1}^{n}(x_j - x_0) \;\; det \begin{bmatrix} 1 & x_1 + x_0 & . & . & . & \sum_{i=0}^{n-1} x_1^{n-1-i} x_0^i \\ 1 & x_2 + x_0 & . & . & . & \sum_{i=0}^{n-1} x_2^{n-1-i} x_0^i \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & x_n + x_0 & . & . & . & \sum_{i=0}^{n-1} x_n^{n-1-i} x_0^i \end{bmatrix}$$

$$= \prod_{j=1}^{n}(x_j - x_0) \;\; det \begin{bmatrix} 1 & x_1 & x_1^2 & . & . & . & x_1^{n-1} \\ 1 & x_2 & x_2^2 & . & . & . & x_2^{n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_1 & x_n^2 & . & . & . & x_n^{n-1} \end{bmatrix} \begin{bmatrix} 1 & x_0 & x_0^2 & . & . & . & x_0^{n-1} \\ 0 & 1 & x_0 & . & . & . & x_0^{n-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{bmatrix}$$

again, by Cauchy, we have that

$$= \prod_{j=1}^{n}(x_j - x_0) \;\; det \begin{bmatrix} 1 & x_1 & x_1^2 & . & . & . & x_1^{n-1} \\ 1 & x_2 & x_2^2 & . & . & . & x_2^{n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_n & x_n^2 & . & . & . & x_n^{n-1} \end{bmatrix}$$

$$= \prod_{j=1}^{n}(x_j - x_0) \qquad \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

$$= \prod_{0 \leq i < j \leq n} (x_j - x_i)$$

(e) Compare the coefficient of $x_1 \; x_2^2 \; x_3^3 \; ... \; x_n^n$ to obtain the value of C

Solution: The coefficient of the leading diagonal, $\prod_{n} x_n^n$ is equals to 1 in both the determinant and the product, the value of $C = 1$.

2. Monic Legendre polynomials on $[-1, 1]$ are defined as follows:

$$q_0(x) = 1$$

$$q_1(x) = x$$

and $q_n(x)$ is a monic polynomial of degree n such that $\int_{-1}^{1} q_n(x) q_m(x) dx = 0$ for all $m \neq n$.

(a) Show that these orthogonal polynomials satisfy

$$q_{n+1}(x) = x q_n(x) - \left(\frac{n^2}{4n^2 - 1} \right) q_{n-1}(x)$$

Solution:

let $n = 1$,

$$\begin{aligned} q_2(x) &= xq_1(x) - \left(\frac{1}{4-1}\right) q_0(x) \\ &= x(x) - \left(\frac{1}{3}\right)(1) \\ &= x^2 - \left(\frac{1}{3}\right) \\ &= 3x^2 - 1 \end{aligned}$$

(b) Prove that if $p(x)$ is a monic polynomial of degree n minimizing $||p(x)||_2$, then $p(x) = q_n(x)$

Solution:

(c) Conclude that the Legendre nodes (i.e., the roots of the Legendre polynomial) minimize $\int_{-1}^{1} \left(\prod_{k=0}^{n}(x - x_k)\right)^2 dx$

Solution:

3. The Chebyshev polynomials of the first kind are defined as

$$T_n(x) = cos(n\ arccos(x))$$

(a) Show that the Chebyshev polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0$$

(b) Show that the Chebyshev polynomials of the first kind satisfy the recurrence:

$$T_{n+1} = 2xT_n - T_{n-1}$$

with $T_0(x) = 1$ and $T_1(x) = x$.

(c) Show that $T_n(x)$ is a polynomial of degree n with leading coefficient as 2^{n-1} for $n \geq 1$

(d) All zeros of $T_{n+1}(x)$ are in the interval $[-1, 1]$ and given by $x_k = cos\left(\frac{2k+1}{2n+2}\pi\right)$, where $k \in \{0, 1, 2, ..., n\}$

(e) Conclude that $T_n(x)$ alternates between $+1$ and -1 exactly $n + 1$ times

(f) Show that

$$\left|\prod_{k=0}^{n}(x - x_k)\right| \leq \frac{1}{2^n}$$

for all $x \in [-1, 1]$

(g) For any choice of nodes $\{y_k\}_{k=0}^{n}$, consider the polynomial $P_{n+1}(x) = \prod_{k=0}^{n}(x - y_k)$ and look at $F(x) = P_{n+1}(x) - \frac{T_{n+1}(x)}{2^n}$

If $|P_{n+1}(x)| \leq \frac{1}{2^n}$, show that $F(x)$ alternates in sign $n + 2$ times on the interval $[-1, 1]$. Hence, conclude that $F(x)$ has to be identically zero and therefore conclude that Chebyshev nodes of the first kind minimizes $max\ x \in [-1, 1] \left|\prod_{k=0}^{n}(x - x_k)\right|$

4. Consider the uniformly spaced nodes, the Legendre nodes, Chebyshev nodes of the first kind for $n+1$ points. For all these sets of points, perform the following:

(a) Plot the condition number of the Vandermonde matrix as a function of n (Use semilogy for plot). Comment on how the condition number scales with n for all these three sets of nodes. You will get three curves (one corresponding to uniform nodes, one corresponding to Legendre nodes and another one corresponding to Chebyshev nodes).

(b) For the Range function obtain the and plot the interpolant by the three set of nodes for $n \in \{5, 7, 9, ..., 33, 35\}$ by

- Solving the linear system
- Using Lagrange polynomials

Comment on the above.

(c) Plot the decay in maximum relative error as a function of n (on a semilogy) plot for the three different interpolants (vary n from 5 to 35 in steps of 2). To get the maximum relative error, evaluate the interpolant and the Runge function at 1001 equally spaced points and compute the maximum relative error at these 1001 points.

5. Show that for any set of interpolant nodes, we have

$$\sum_{j=0}^{n} x_j^m l_j(x) = x^m$$

for all $m \in \{0, 1, ..., n\}$, where $l_j(x)$ is the j^{th} Lagrange polynomial.